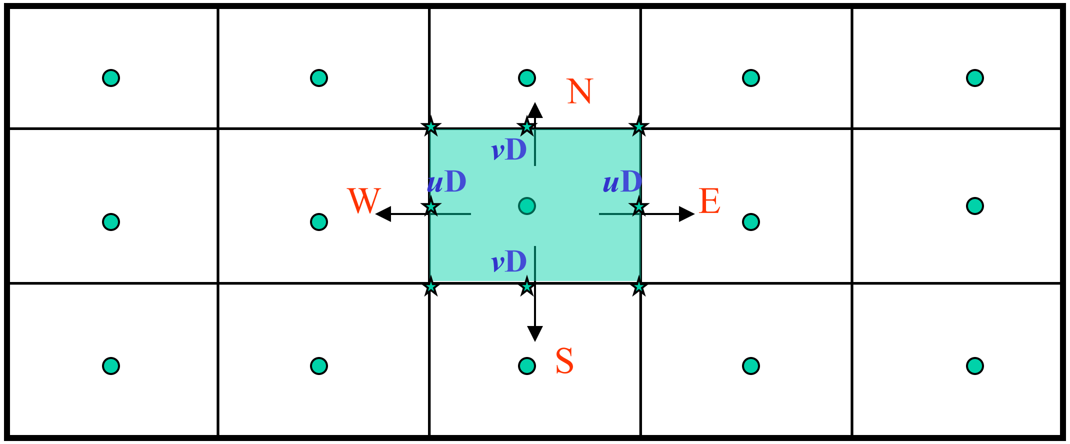
MAR513 Lecture 5: Finite-Volume Methods

Unlike finite-difference and finite-element methods, the computational domain in the finite-volume methods is divided into many control volumes (CV) and the governing equations are solved in its integral form in individual control volumes.

For example:

$$\iint_{\Omega} [\frac{\partial \zeta}{\partial t} + \nabla \cdot (\vec{v}D)] dx\, dy = 0 \Rightarrow \frac{\partial \zeta}{\partial t} = -\frac{1}{\Omega} \oint_{s} v_n D ds \tag{7.1}$$

Structured grids

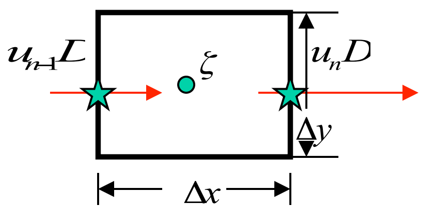


1. Assign the elevation at the center of each rectangular control volume;

2. Define that outflow is positive and inflow is negative;

3. Calculate the net flux

Approximation of volume integrals



$$\iint_{\Omega} \frac{\partial uD}{\partial x} dxdy = \int_S uD\, dy = [u_n D_n - u_{n-1} D_{n-1}]\Delta y$$

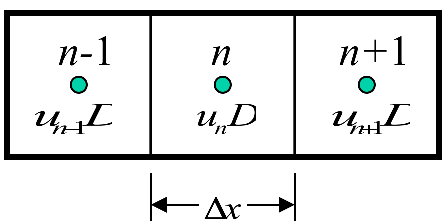
$$\Downarrow$$

$$\frac{\partial uD}{\partial x} \Delta x \Delta y \approx [u_n D_n - u_{n-1} D_{n-1}]\Delta y$$

$$\Downarrow$$

$$\frac{\partial uD}{\partial x} \approx \frac{[u_n D_n - u_{n-1} D_{n-1}]}{\Delta x}$$

The first order upwind scheme



$$\iint_{\Omega} \frac{\partial uD}{\partial x} dxdy = \int_S uD\, dy = [\frac{u_{n+1}D_{n+1} + u_n D_n}{2} - \frac{u_n D_n + u_{n-1}D_{n-1}}{2}]\Delta y$$

$$= \frac{\Delta y}{2}(u_{n+1}D_{n+1} - u_{n-1}D_{n-1})$$

$$\Downarrow$$

$$\frac{\partial uD}{\partial x} \Delta x \Delta y \approx \frac{\Delta y}{2}(u_{n+1}D_{n+1} - u_{n-1}D_{n-1})$$

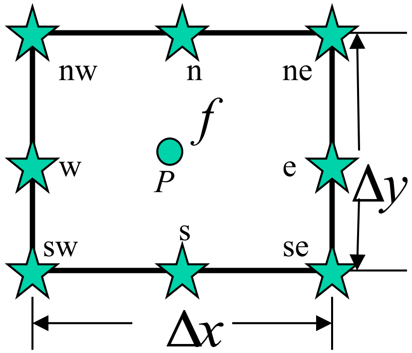
$$\Downarrow$$

$$\frac{\partial uD}{\partial x} \approx \frac{u_{n+1}D_{n+1} - u_{n-1}D_{n-1}}{2\Delta x}$$

The second order central scheme

Consider an arbitrary function like

$$F = \oint f \, ds$$



On the east side, for the first order approximation,

$$F_e = f_e \Delta y$$

For the second order approximation,

$$F_e = \frac{1}{2}(f_{se} + f_{ne})\Delta y$$

For the fourth order approximation,

$$F_e = \frac{\Delta y}{6}(f_{se} + 4 f_e + f_{ne})$$

Consider an arbitrary function like

$$F = \iint_{\Omega} f d\Omega = \bar{f} \Delta x \Delta y$$

For the first order approximation

$$F = f_P \Delta x \Delta y$$

For the second order approximation,

$$F = \bar{f} \Delta x \Delta y$$

The fourth order approximation can be obtained by using the bi-quadratic shape funcion:

$$f(x,y) = a_o + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy + a_6 x^2 y + a_7 xy^2 + a_8 x^2 y^2$$

Need 9 coefficients,which can determined by fitting the function to the value of *f* at *9* locations (nw,w,sw, n, *p*,s, ne,e,and se).

$$F = \Delta x \Delta y [a_o + \frac{a_3}{12}(\Delta x)^2 + \frac{a_4}{12}(\Delta y)^2 + \frac{a_8}{144}(\Delta x)^2 (\Delta y)^2]$$

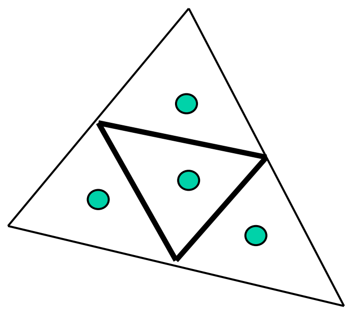
$$= \frac{\Delta x \Delta y}{36}(16 f_p + 4 f_s + 4 f_n + 4 f_w + 4 f_e + f_{se} + f_{sw} + f_{ne} + f_{nw})$$

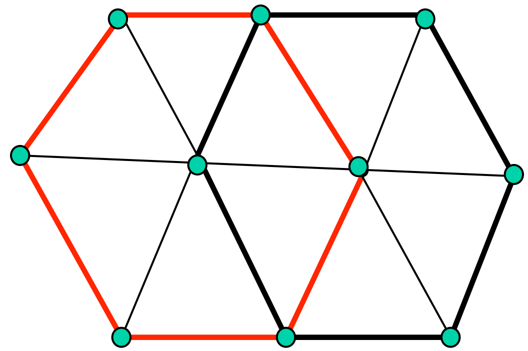
For the cell-centered grids, the value at P point is known,but values at other points must be obtained by interpolation from surrounding cell-centered nodes.

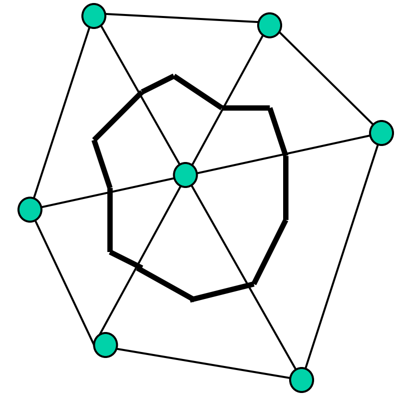
Comments;

Structured grid finite-volume model is a special type of the finite-difference methods and they always can convert from one to another. So, little efforts need to make to convert a finite-difference model to a finite-volume model under structured grids.

3. Popular unstructured triangular FVM grid in CFD:





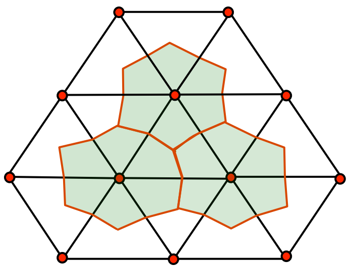


1. **Cell-centered**
2. **Cell-vertex overlapping**
3. **Cell-vertex median**

Characteristics of the oceanic motion:

- Free surface----How to calculate accurately the integral form of the surface pressure gradient forcing?
- Steep bottom topography----How to ensure the mass conservation in a two mode model system?
- Open boundary conditions----How to minimize the wave energy reflection at open boundaries?

Cell vertex median grid



A Grid: All variables (ζ,u,v, ω, θ, s..) at nodes

Advantage:

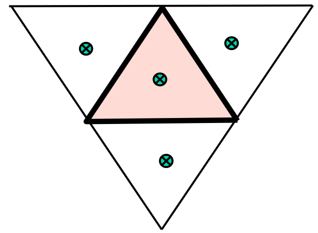
1) Simple
2) Guarantee the mass conservation for tracers

Disadvantage:

The accuracy of the surface elevation gradient forcing is sensitive to the shape of the control element (due to interpolation)

Hard to ensure the mass conservation at open boundaries

Cell-centered



B Grid: All variables (ζ,u,v, ω, θ, s..) at centroids

Advantage:

1) Simple
2) Better to advection calculation

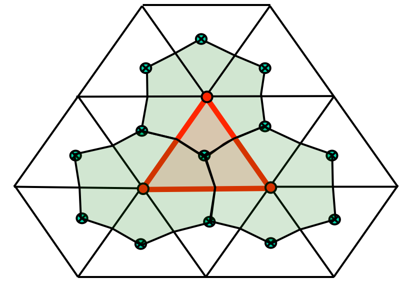
Disadvantage:

Hard to guarantee The accuracy of the surface elevation gradient forcing

Hard to ensure the mass conservation at open boundaries

Hard to ensure the mass conservation for tracer calculation

Cell-vertex-centered



C Grid: ● ζ, ω, θ, s, K_m, K_h…

Advantage:

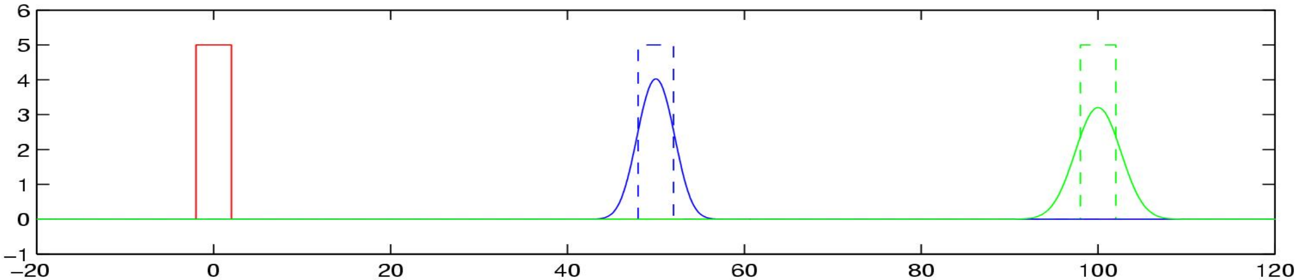
- Combine the best of A and B Grids;
- Easy to ensure the mass conservation for tracers
- Easy to introduce the mass conservative open boundary conditions

FVCOM

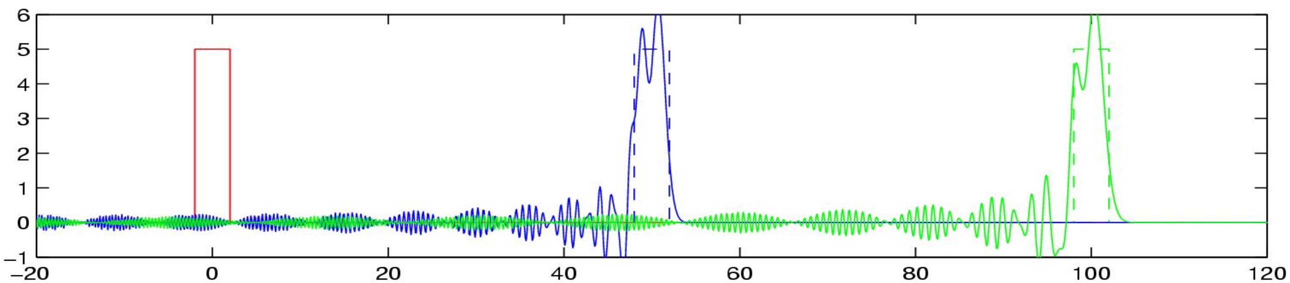
1. Advection Scheme

$$\frac{\partial F}{\partial t} + C\frac{\partial F}{\partial x} = 0 \qquad F(x,0) = \begin{cases} 5 & -2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \quad C = 1$$

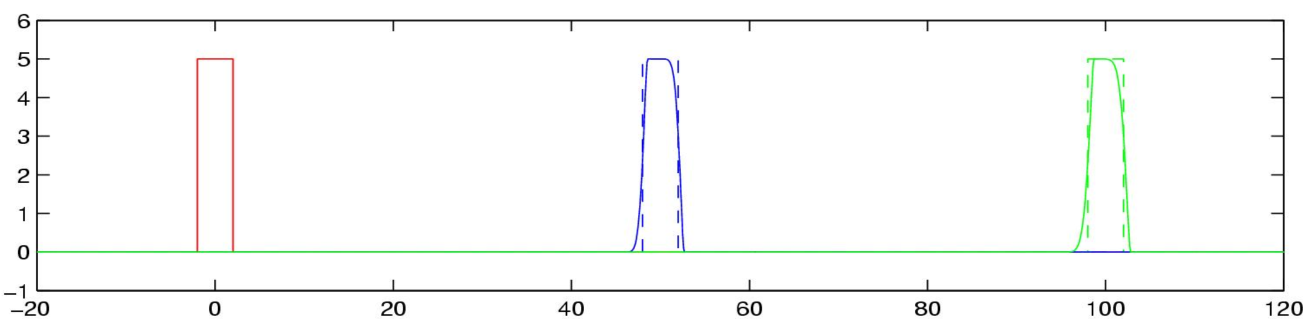
$\Delta t = 0.05$, $\Delta x = 0.1$



Upwind finite-difference stream

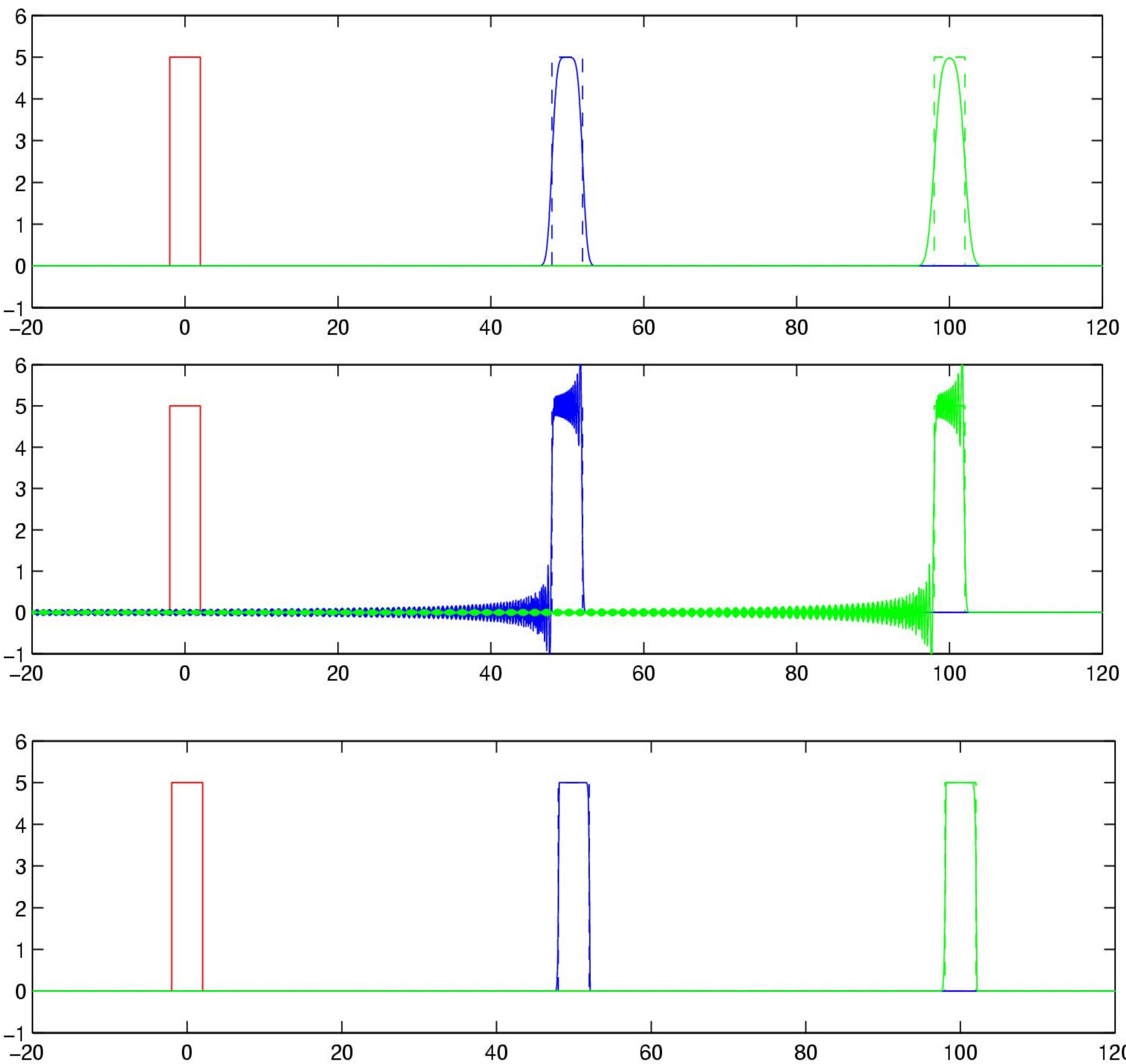


Central finite-difference stream



FVCOM finite-volume flux scheme

$\Delta t = 0.005$,
$\Delta x = 0.01$



Upwind finite-difference stream

Central finite-difference stream

FVCOM finite-volume flux scheme

Wind-induced oscillation

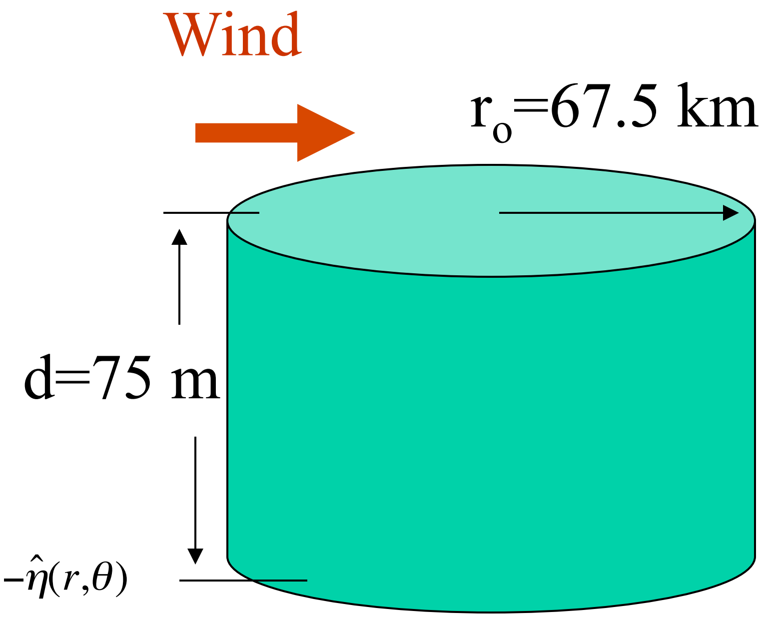
Wind is suddenly imposed at initial

Linear, non-dimensional equations:

$$\frac{\partial u}{\partial t} - v = -\lambda \frac{\partial \zeta}{\partial r}$$

$$\frac{\partial v}{\partial t} + u = -\lambda \frac{\partial \zeta}{r \partial \theta}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\lambda}{r}\left[\frac{\partial (ru)}{\partial r} + \frac{\partial v}{\partial \theta}\right] = 0$$



where $\lambda = \frac{\sqrt{gd}}{r_o f}; \zeta = \eta - \hat{\eta}; \hat{\eta} = \frac{\tau_o r \cos\theta}{\lambda^4}; \tau_o = \frac{g\tau}{r_o^3 f^4}$

and $u|_{r=1} = 0; (u, v, \zeta)_{r=0} \to finite; u|_{t=0} = v|_{t=0} = 0; \zeta|_{t=0} = -\hat{\eta}(r, \theta)$

Solution:

$$\eta(r,\theta,t) = \frac{\tau_o}{\lambda^4}[A_o(r)\cos\theta + \sum_{k=1}^{\infty} a_k A_k(r)\cos(\theta - \sigma_k t)]$$

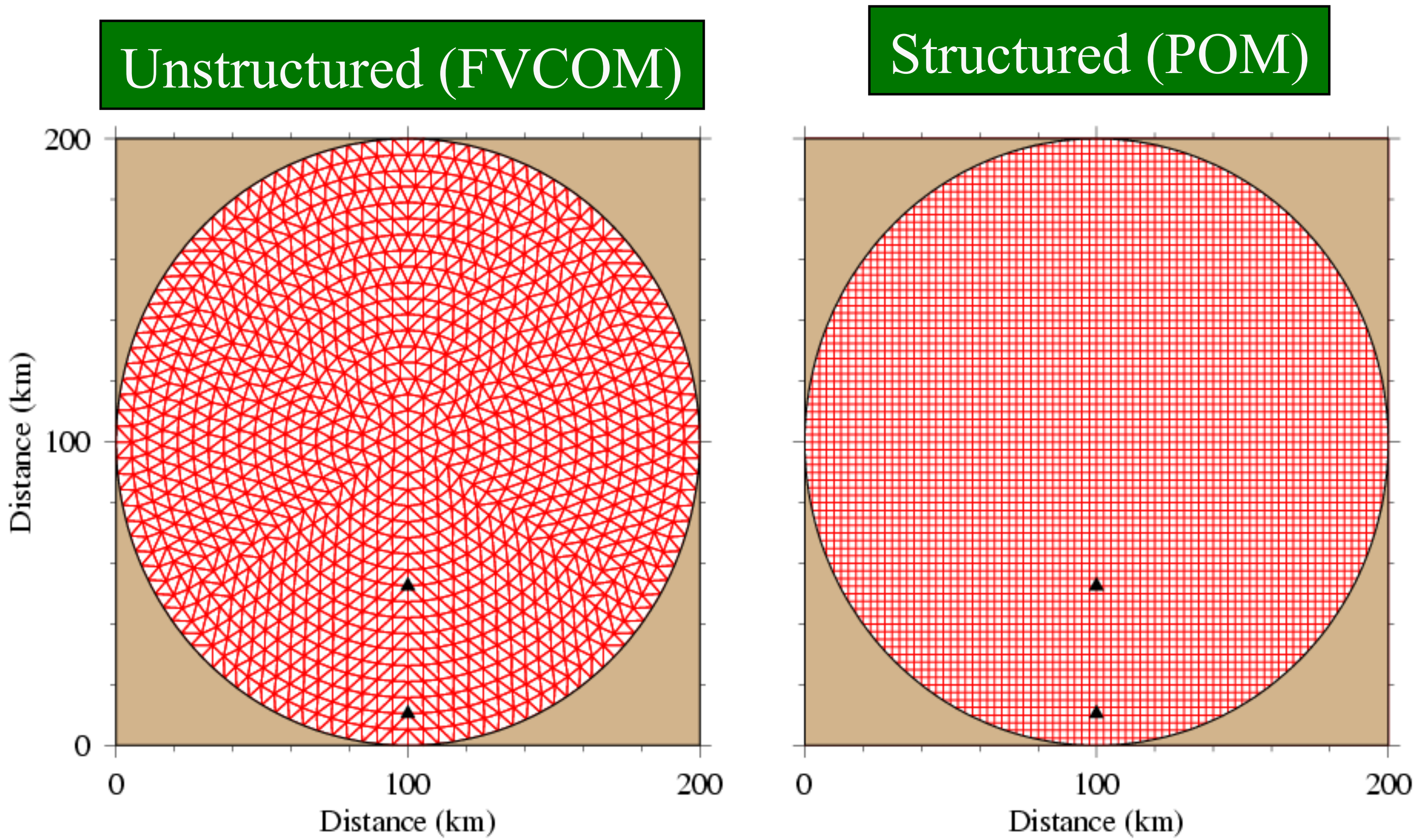
$$u(r,\theta,t) = \frac{\tau_o}{\lambda^3}[(\frac{A_o(r)}{r} - 1)\sin\theta - \sum_{k=1}^{\infty} b_k F_k(r)\sin(\theta - \sigma_k t)]$$

$$v(r,\theta,t) = \frac{\tau_o}{\lambda^3}[(\frac{dA_o(r)}{dr} - 1)\cos\theta - \sum_{k=1}^{\infty} b_k G_k(r)\cos(\theta - \sigma_k t)]$$

Reference:

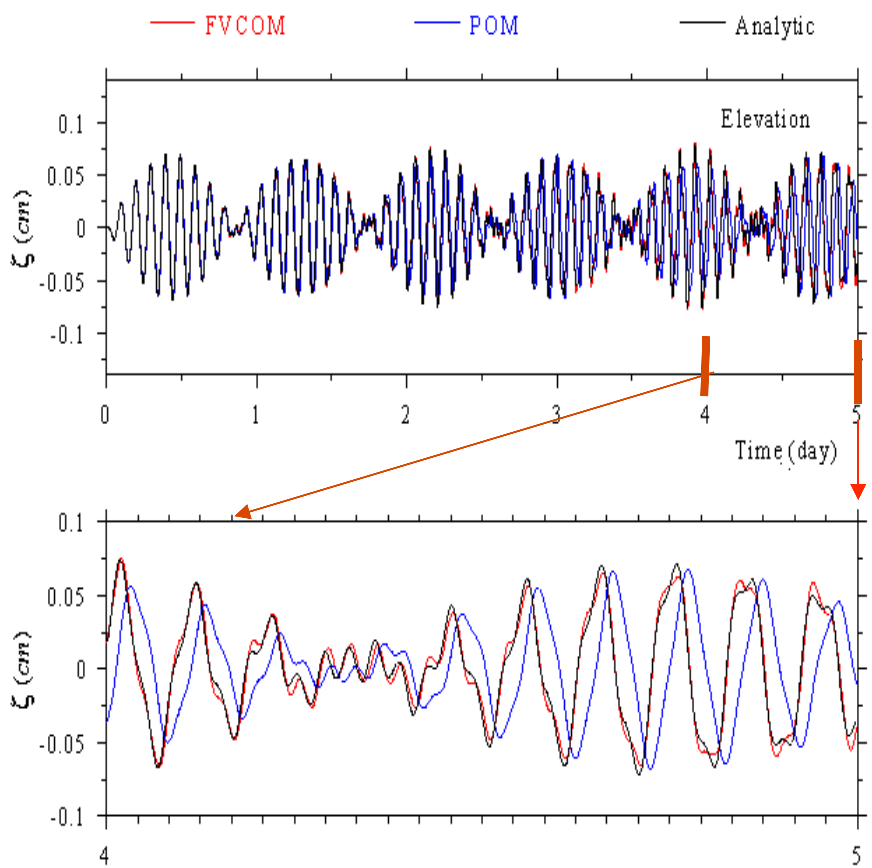
Csanady (1968)

Birchfield (1969)

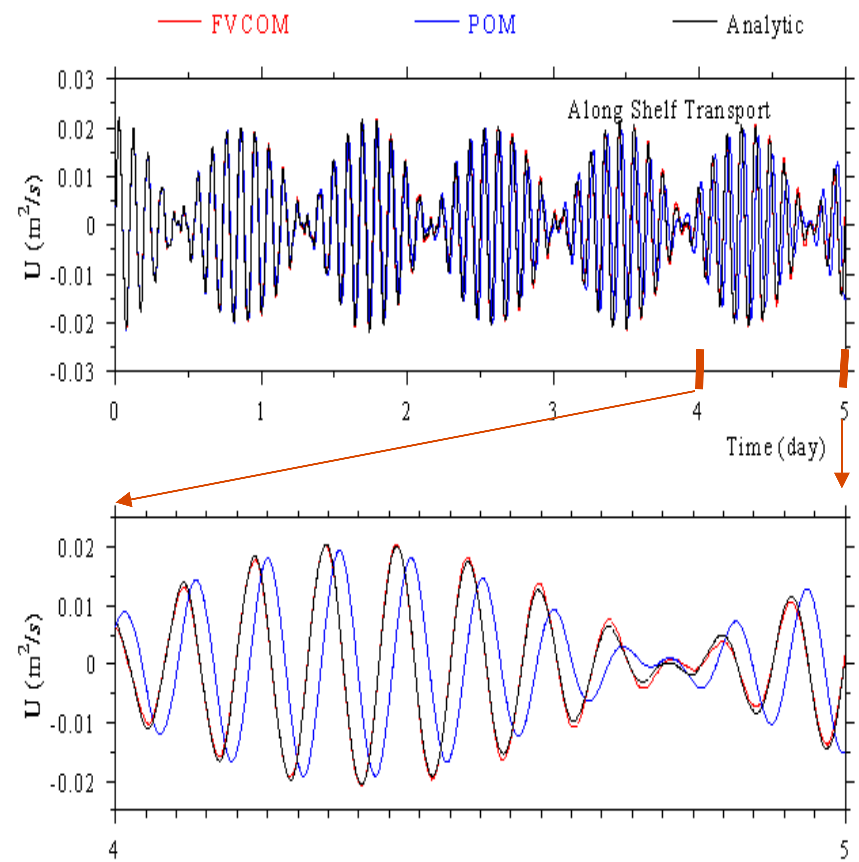


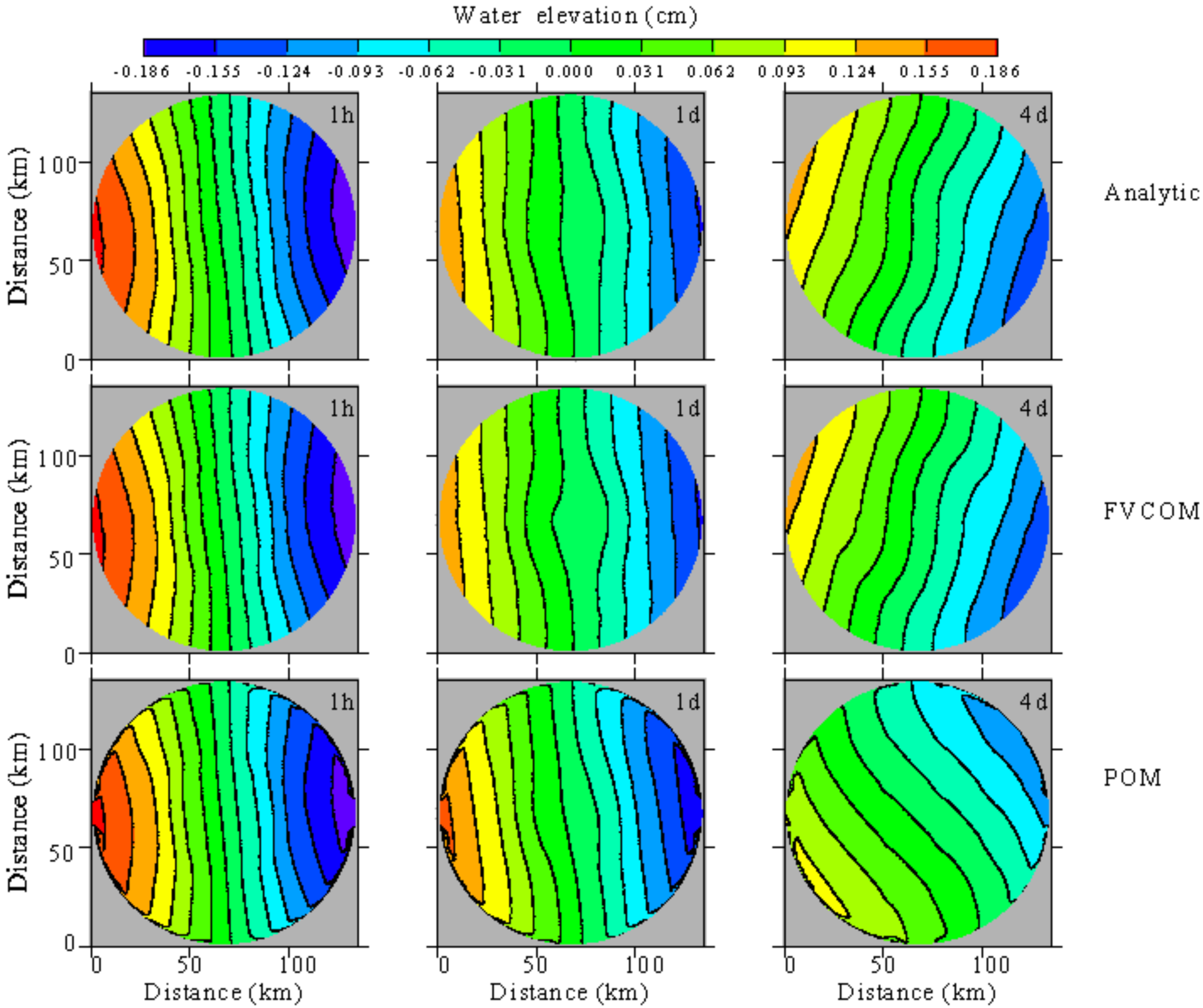
Unstructured (FVCOM)
Structured (POM)
200
100
0
Distance (km)
0
100
200
Distance (km)
0
100
200
Distance (km)

Water elevation

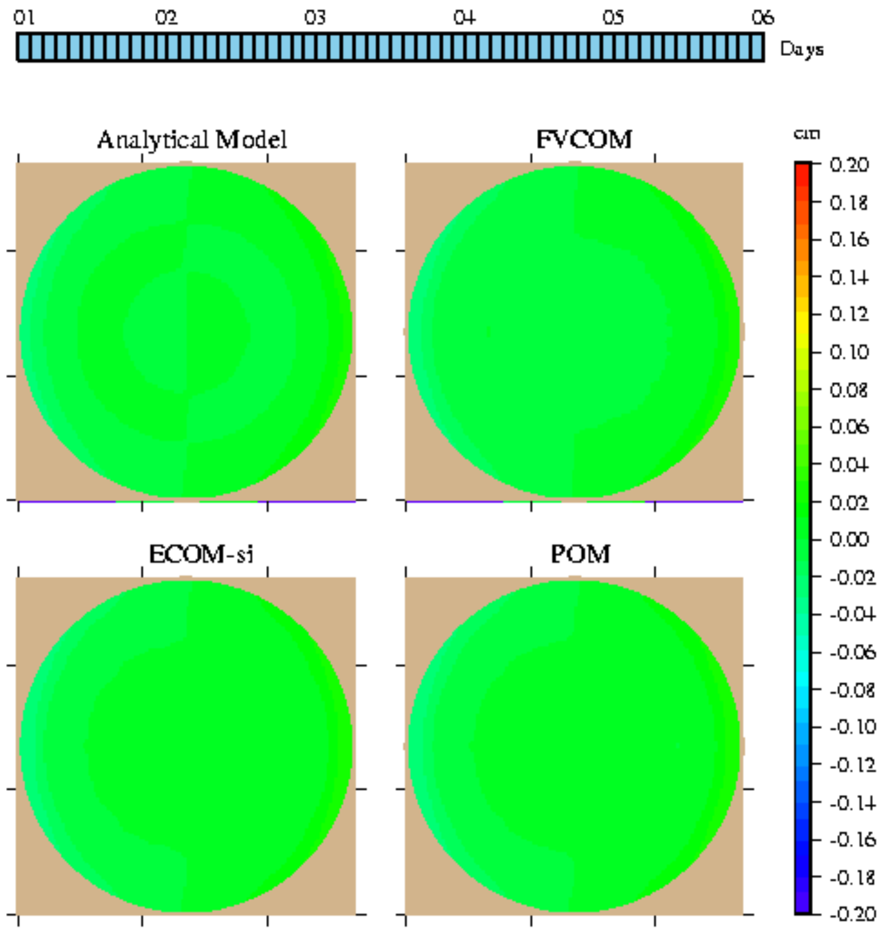


Alongshore transport





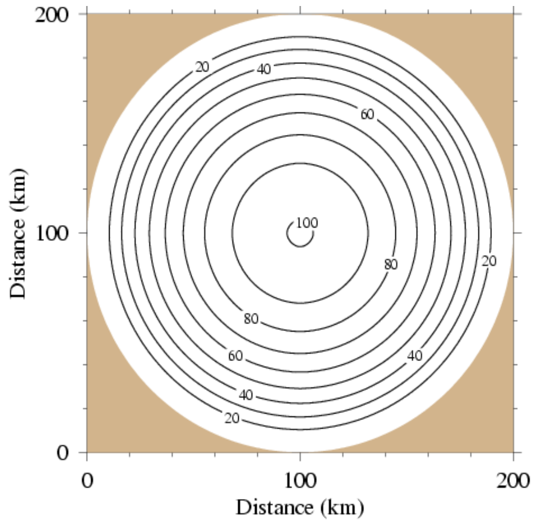
Water elevation (cm)
-0.186 -0.155 -0.124 -0.093 -0.062 -0.031 0.000 0.031 0.062 0.093 0.124 0.155 0.186
1h
1d
4d
Analytic
FVCOM
POM
Distance (km)
0
50
100

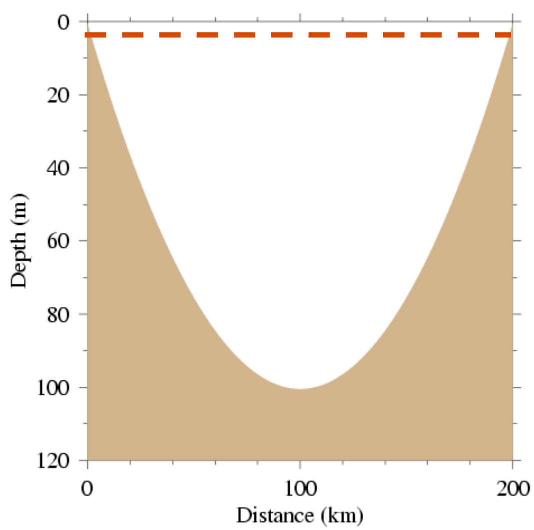


01
02
03
04
05
06
Days
Analytical Model
FVCOM
ECOM-si
POM
cm
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
-0.02
-0.04
-0.06
-0.08
-0.10
-0.12
-0.14
-0.16
-0.18
-0.20

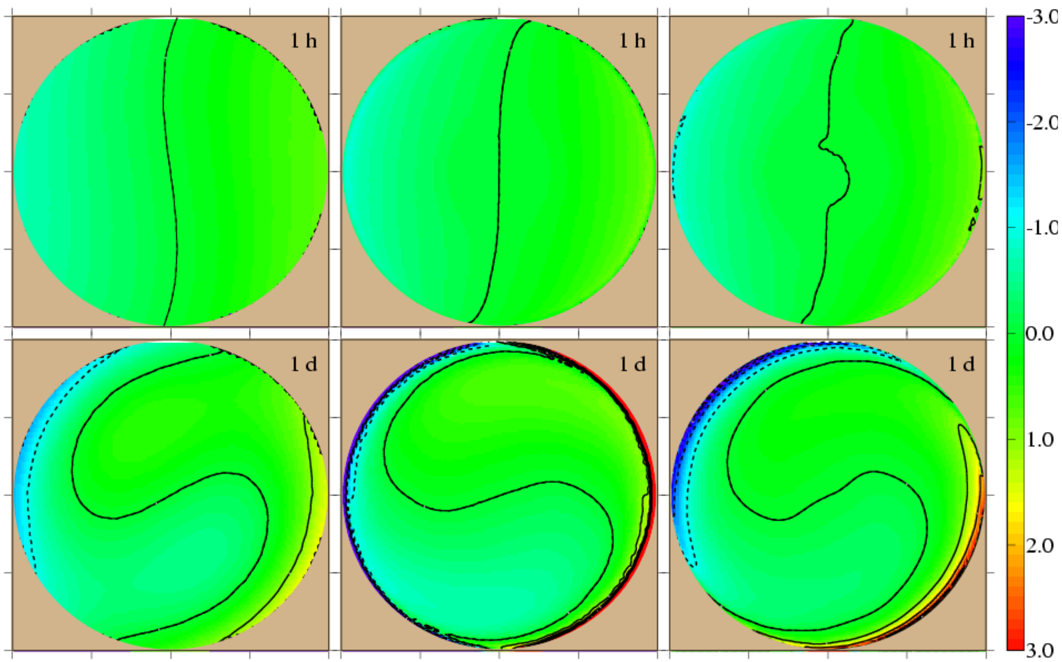
Radial mode: k=1, 2: gravity waves, k=3: topographic wave

Birchfield and Hickie (1977) JPO





Analytical | FVCOM (5 km) | POM (2.5 km)



1 h

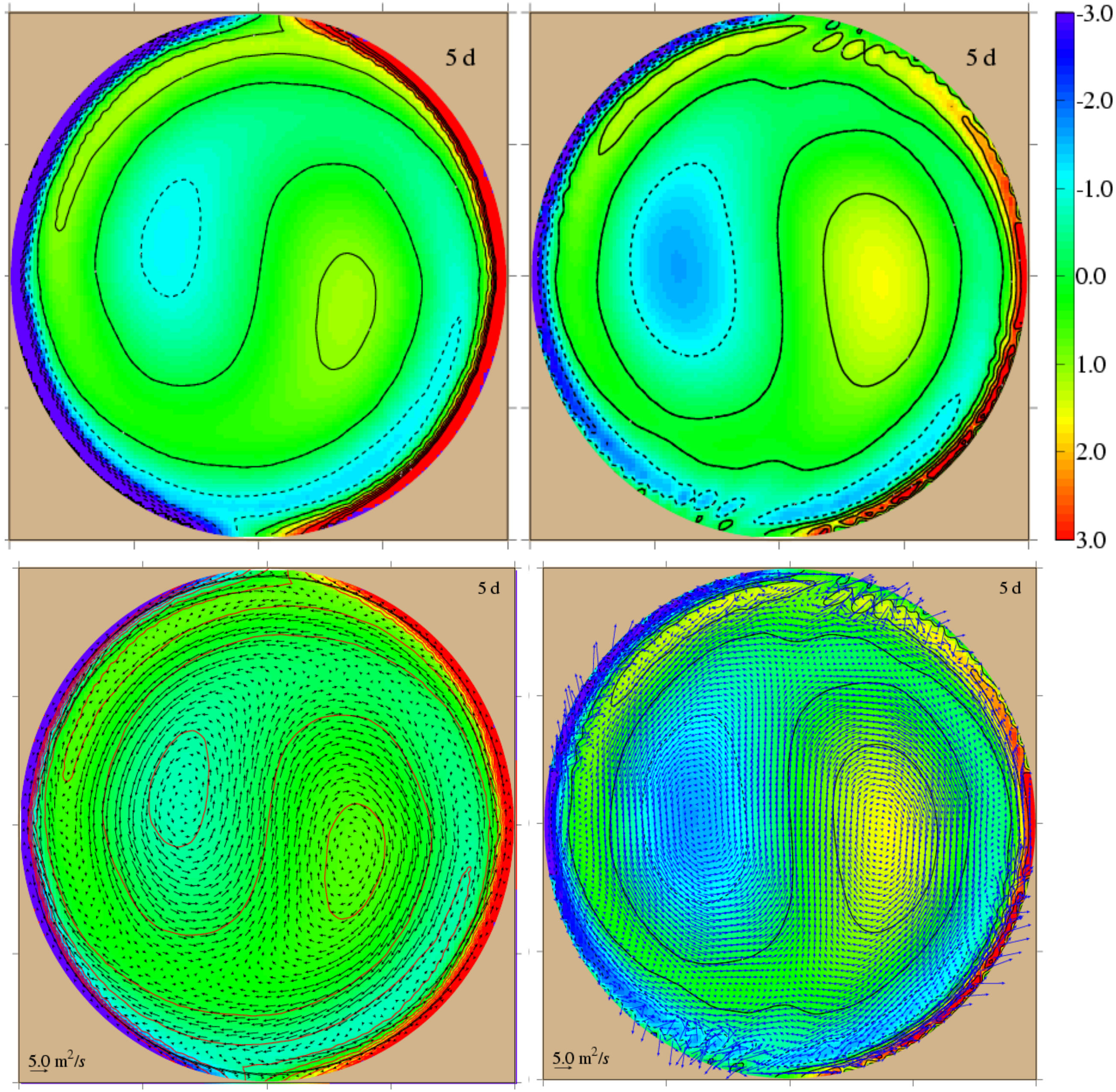
1 d

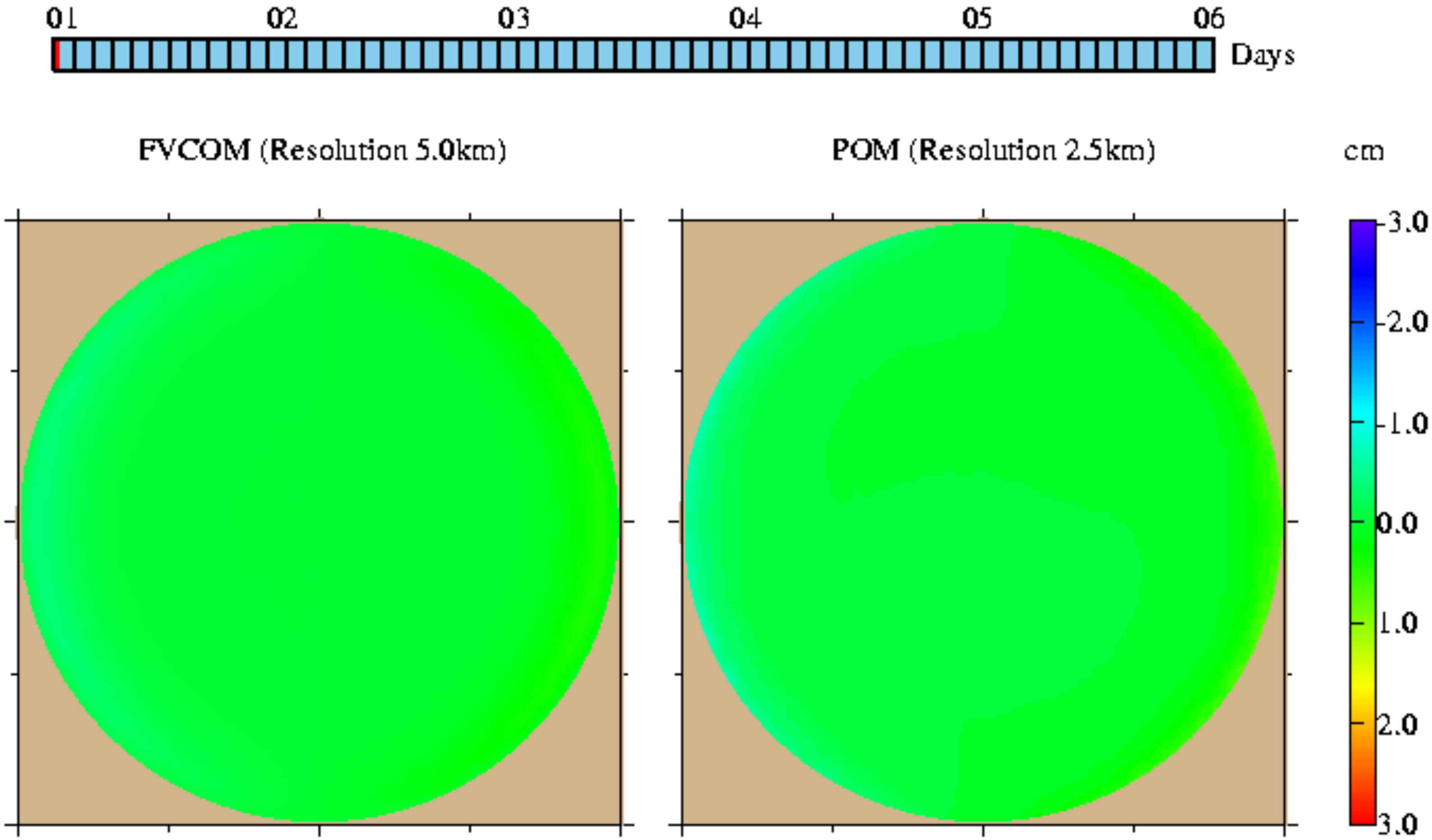
5 days

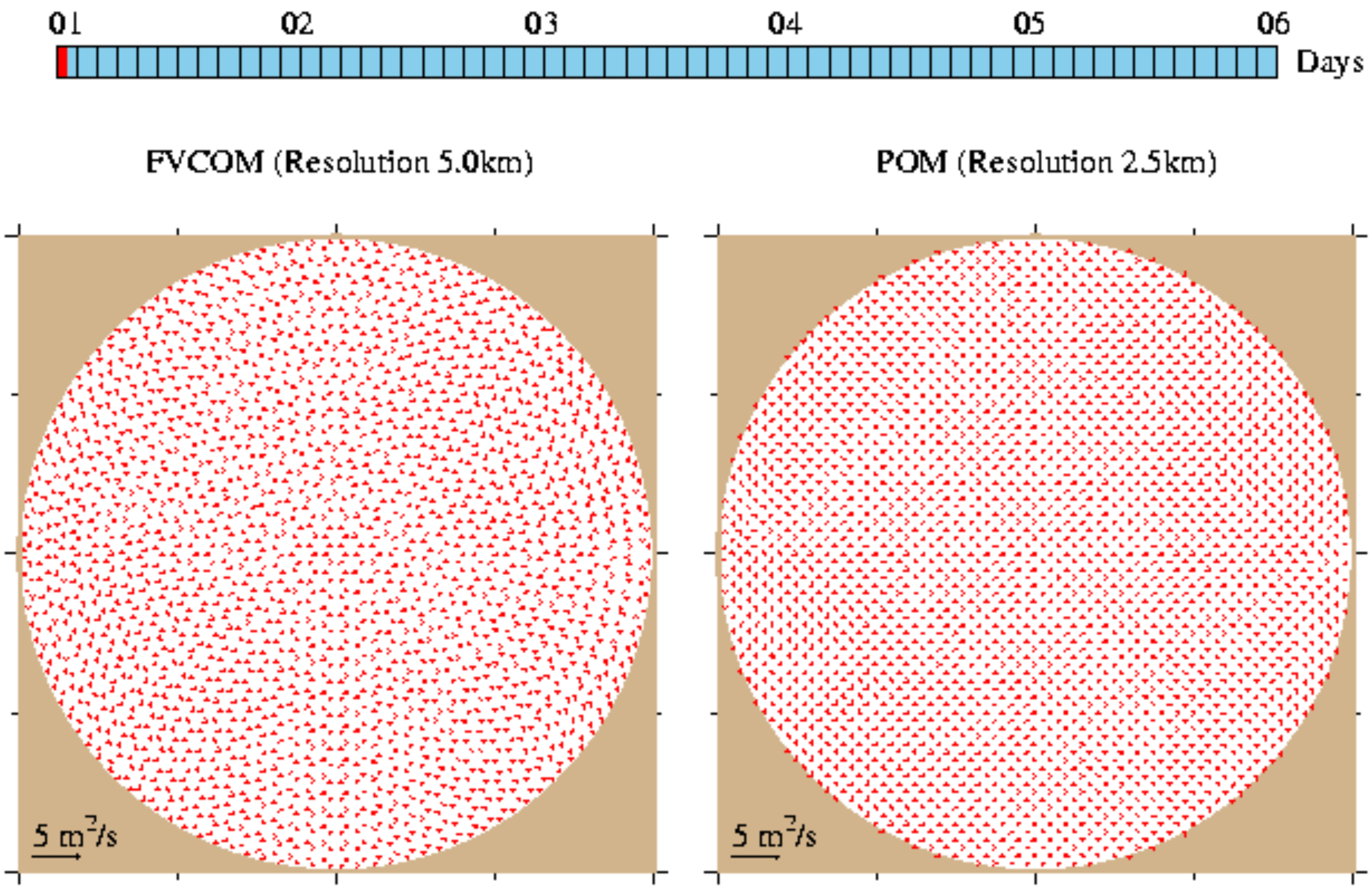
FVCOM (5 km) POM (2.5 km)

ζ: Elevation

$\vec{V}$: Current

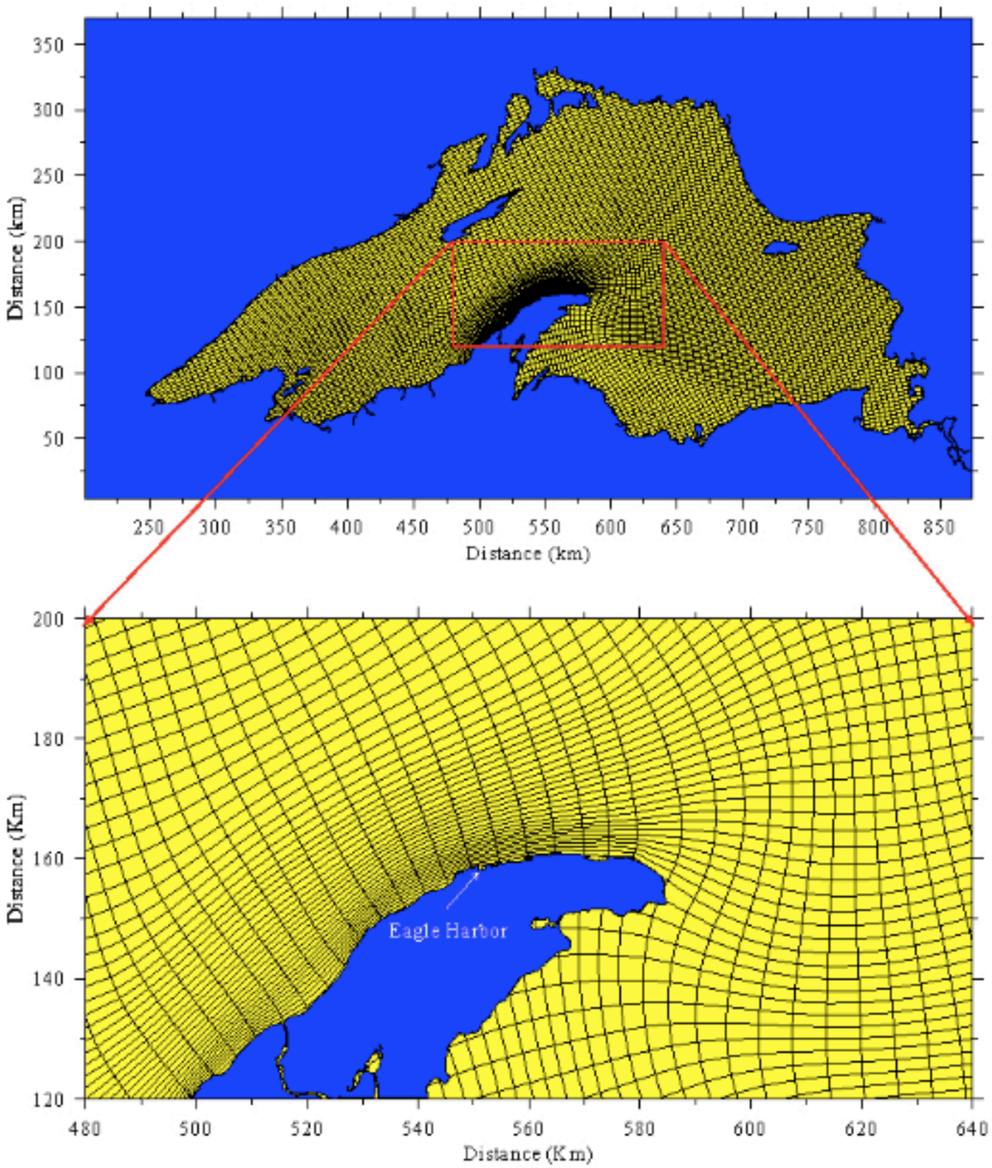



01
02
03
04
05
06
Days
FVCOM (Resolution 5.0km)
POM (Resolution 2.5km)
cm
3.0
2.0
1.0
0.0
1.0
2.0
3.0

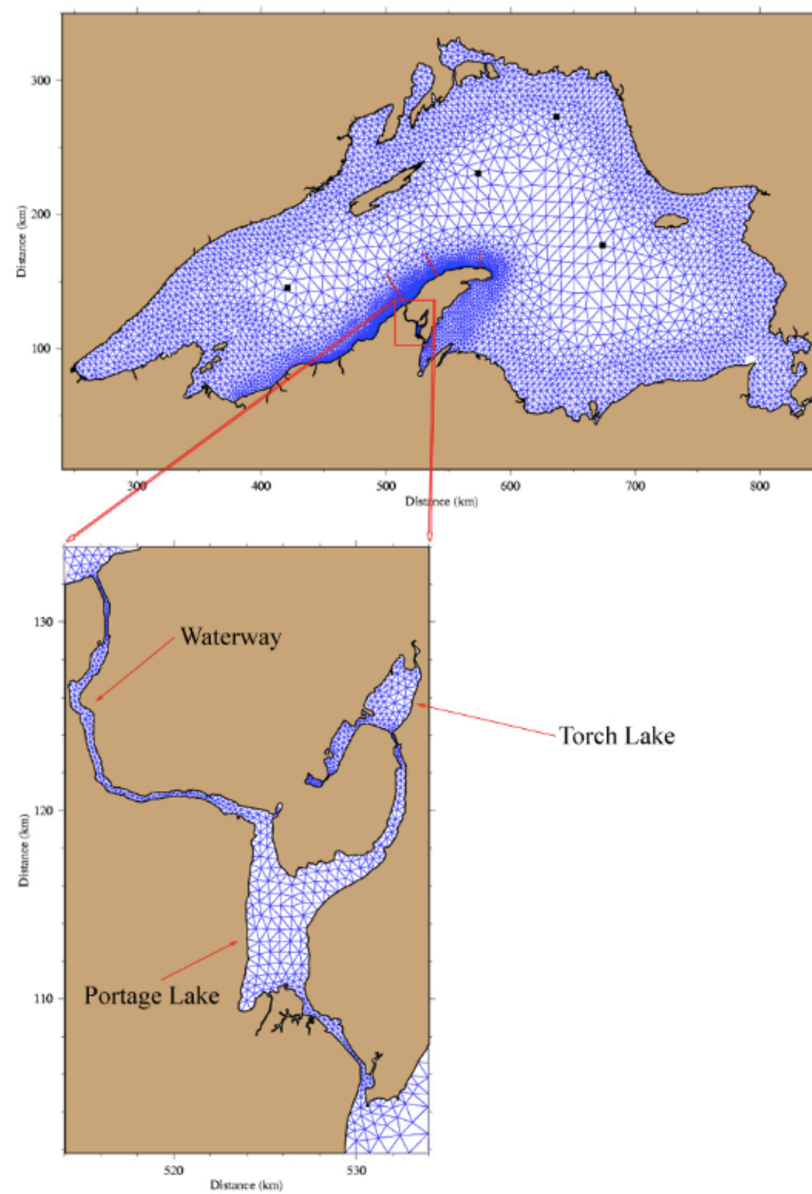


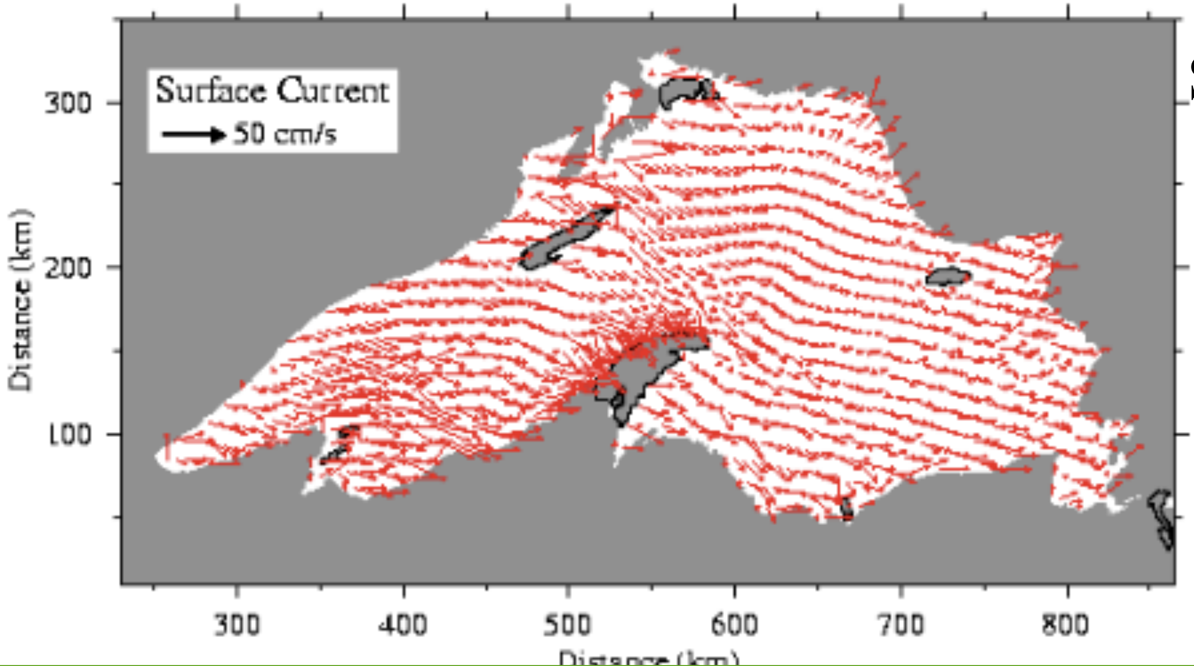
01
02
03
04
05
06
Days
FVCOM (Resolution 5.0km)
POM (Resolution 2.5km)
5 m^2/s
5 m^2/s

Structured (POM)



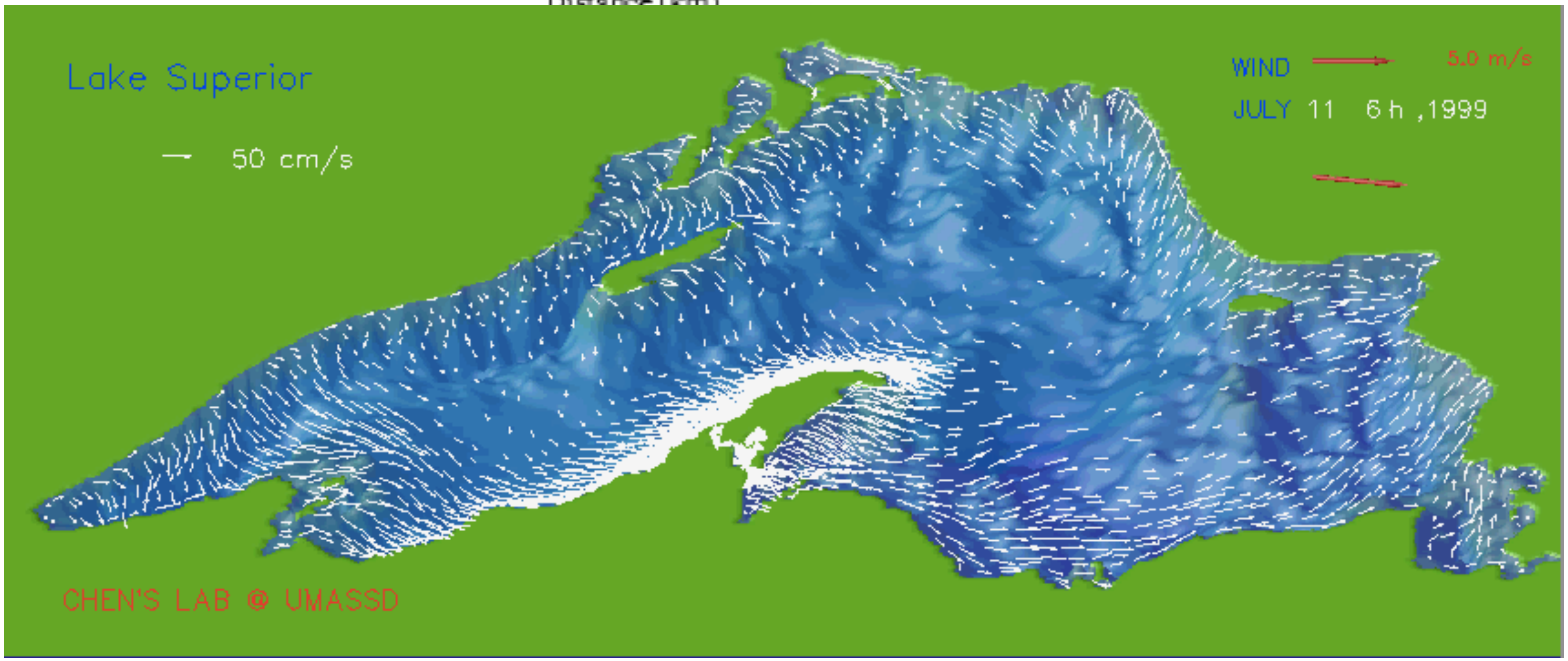
Unstructured (FVCOM)





Structured (POM)

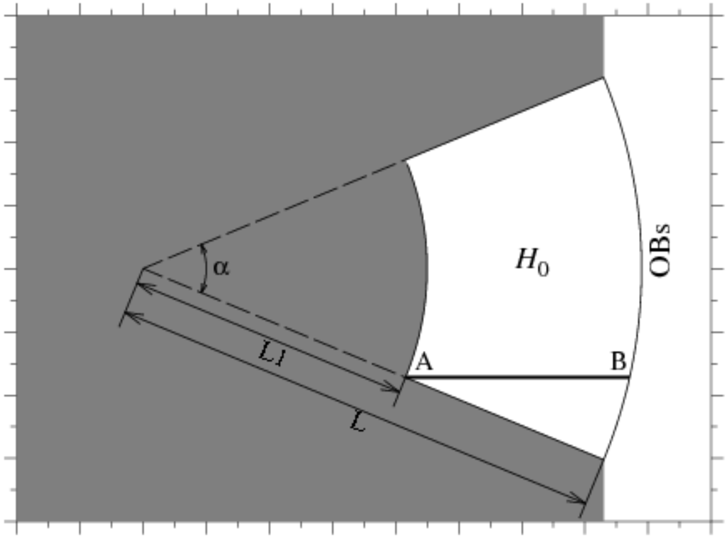
Unstructured (FVCOM)



Tidal Resonance in A Semi-closed Channel

Consider a 2-D linear, non-rotated initial problem such as

$$
\begin{cases}
\dfrac{\partial V_r}{\partial t} + g\dfrac{\partial \eta}{\partial r} = 0 \qquad \dfrac{\partial V_\theta}{\partial t} + g\dfrac{\partial \eta}{r\partial \theta} = 0 \\[2ex]
\dfrac{\partial \eta}{\partial t} + \dfrac{\partial r V_r H_0}{r\partial r} + \dfrac{\partial V_\theta H_0}{r\partial \theta} = 0
\end{cases}
$$



The solution:

$$\eta_0(r,\theta) = [c_1 J_{\gamma_m}(r\frac{\omega}{\sqrt{gH_0}}) + c_2 Y_{\gamma_m}(r\frac{\omega}{\sqrt{gH_0}})]\cdot\cos[\frac{m\pi(\theta+\alpha/2)}{\alpha}]$$

where

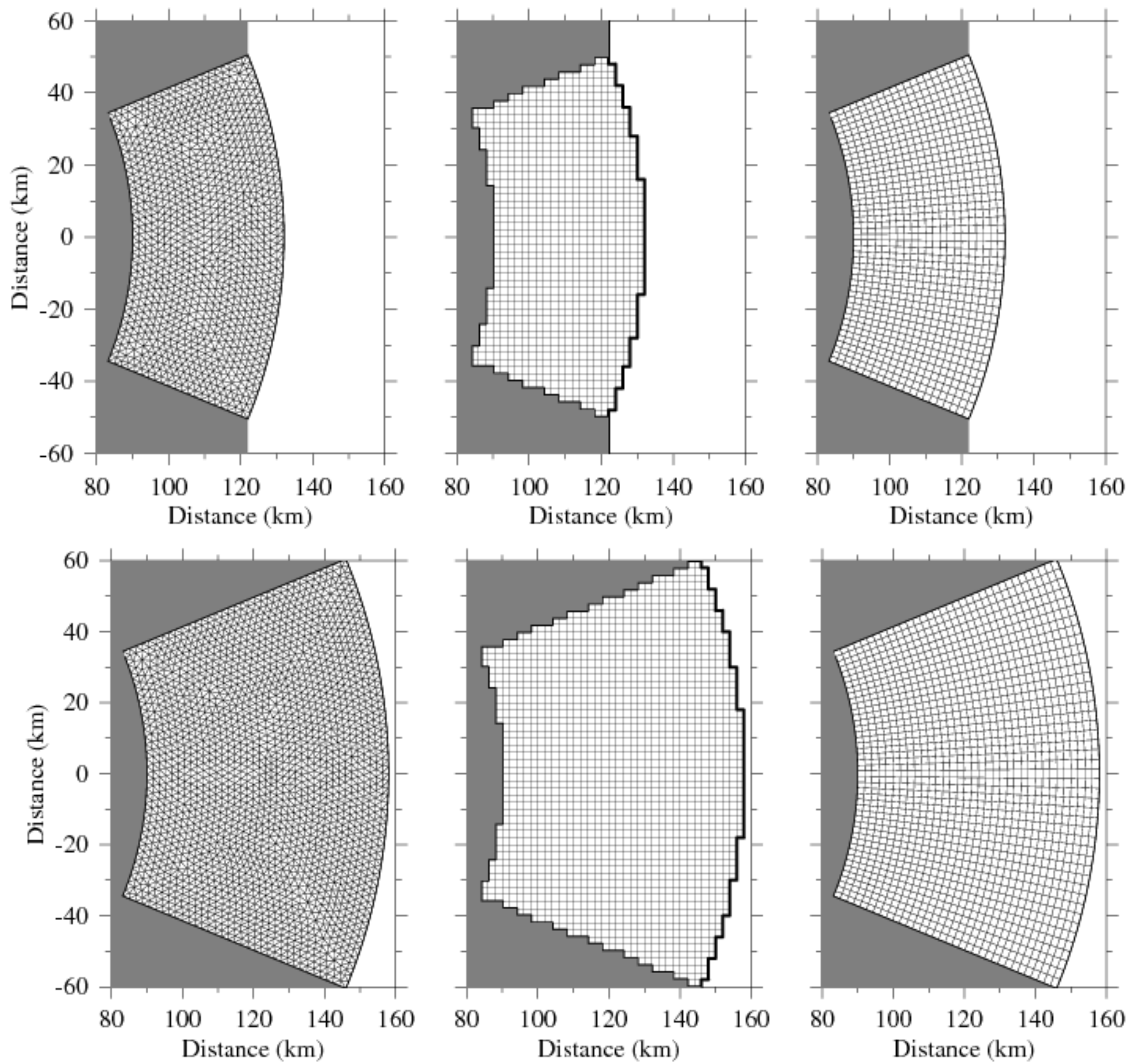
$$c_1 = A\cdot Y'_{\gamma_m}(L_1\frac{\omega}{\sqrt{gH_0}})/[J_{\gamma_m}(L\frac{\omega}{\sqrt{gH_0}})Y'_{\gamma_m}(L_1\frac{\omega}{\sqrt{gH_0}}) - J'_{\gamma_m}(L_1\frac{\omega}{\sqrt{gH_0}})Y_{\gamma_m}(L\frac{\omega}{\sqrt{gH_0}})$$

$$c_2 = -A\cdot J'_{\gamma_m}(L_1\frac{\omega}{\sqrt{gH_0}})/[J_{\gamma_m}(L\frac{\omega}{\sqrt{gH_0}})Y'_{\gamma_m}(L_1\frac{\omega}{\sqrt{gH_0}}) - J'_{\gamma_m}(L_1\frac{\omega}{\sqrt{gH_0}})Y_{\gamma_m}(L\frac{\omega}{\sqrt{gH_0}})$$

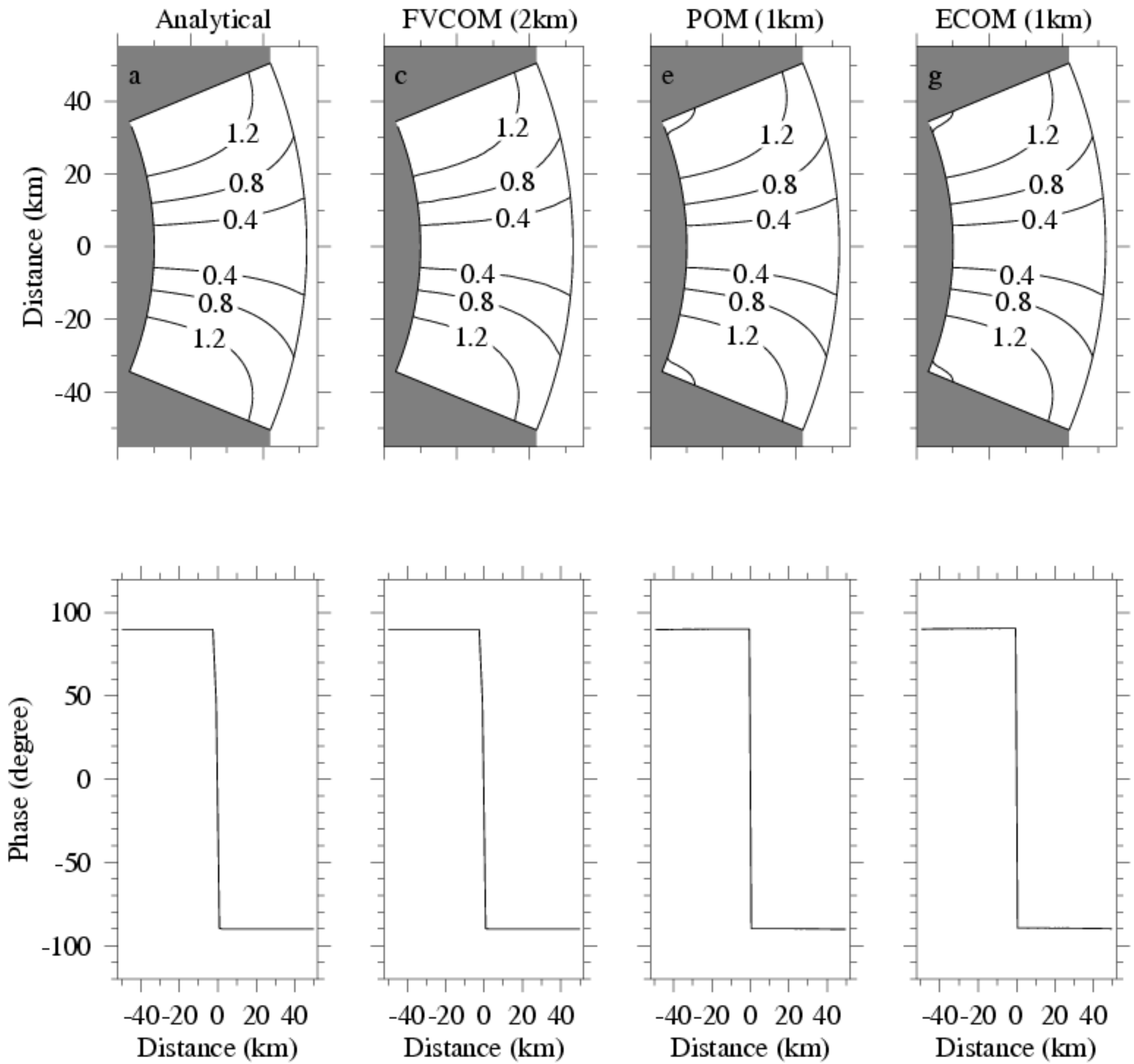
$$\gamma_m = m\pi/\alpha$$

1. Normal condition (non-resonance)

2. Near-resonance condition

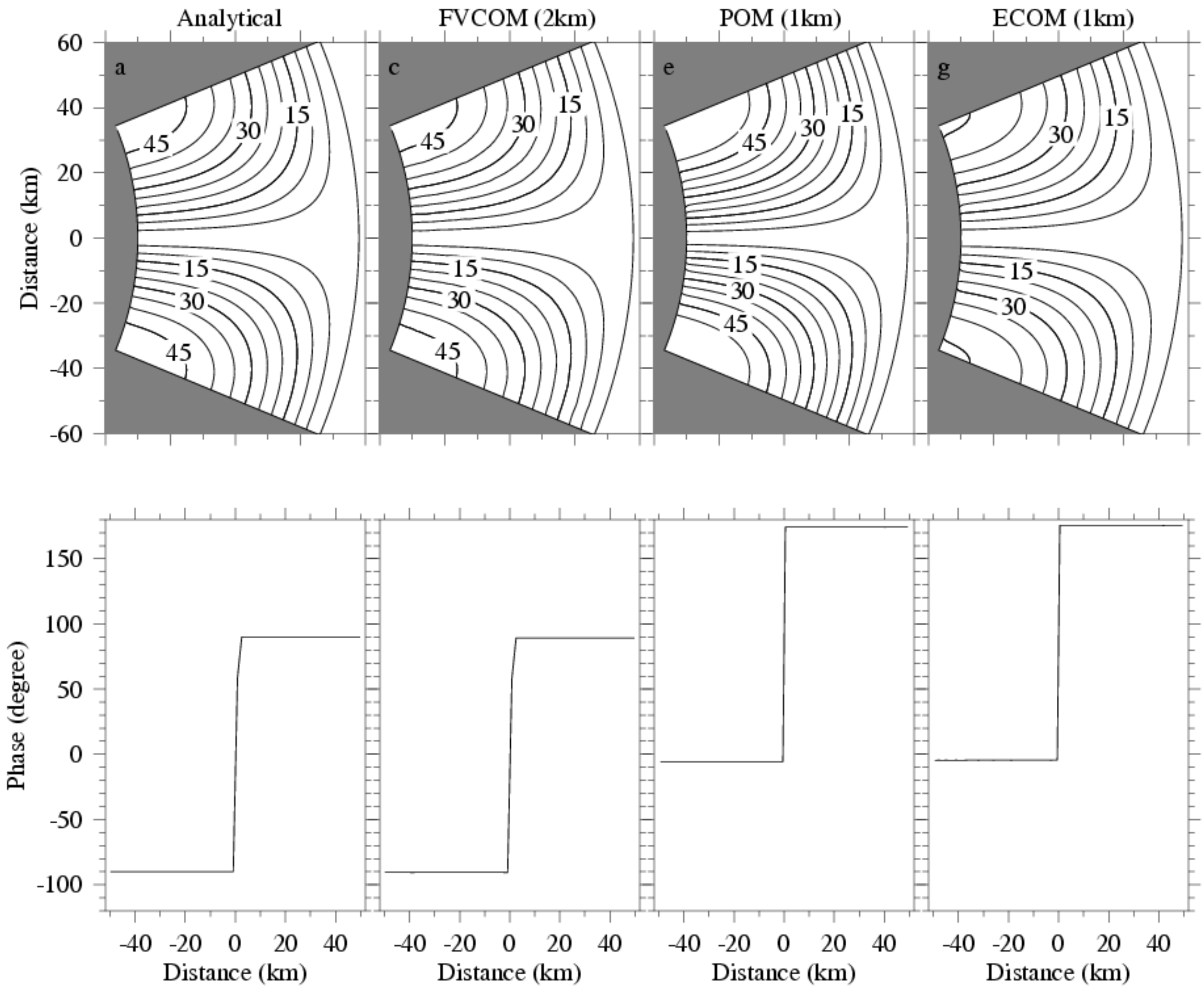


Normal Case

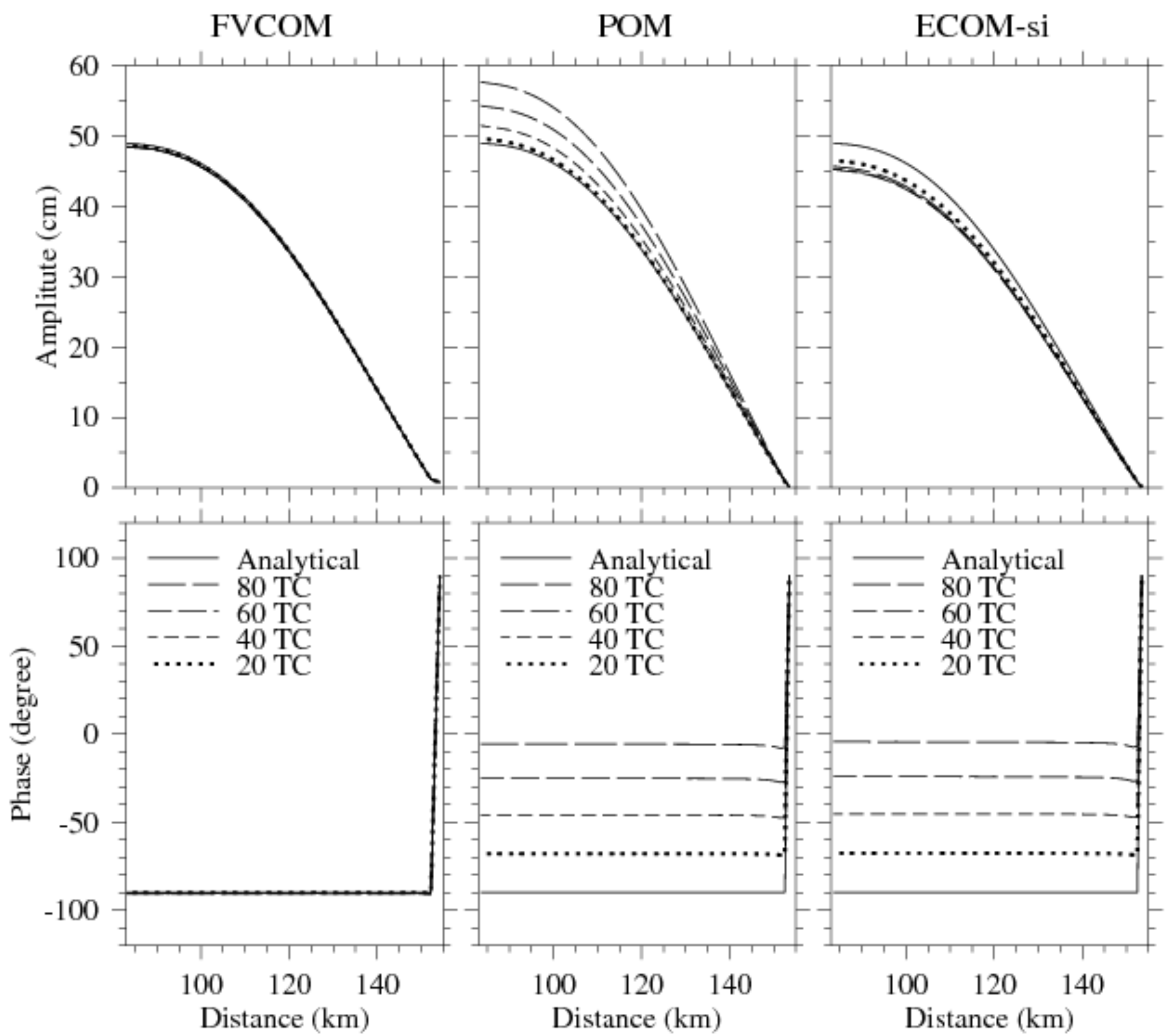


Analytical
FVCOM (2km)
POM (1km)
ECOM (1km)
a
c
e
g
1.2
0.8
0.4
Distance (km)
Phase (degree)
-40-20 0 20 40

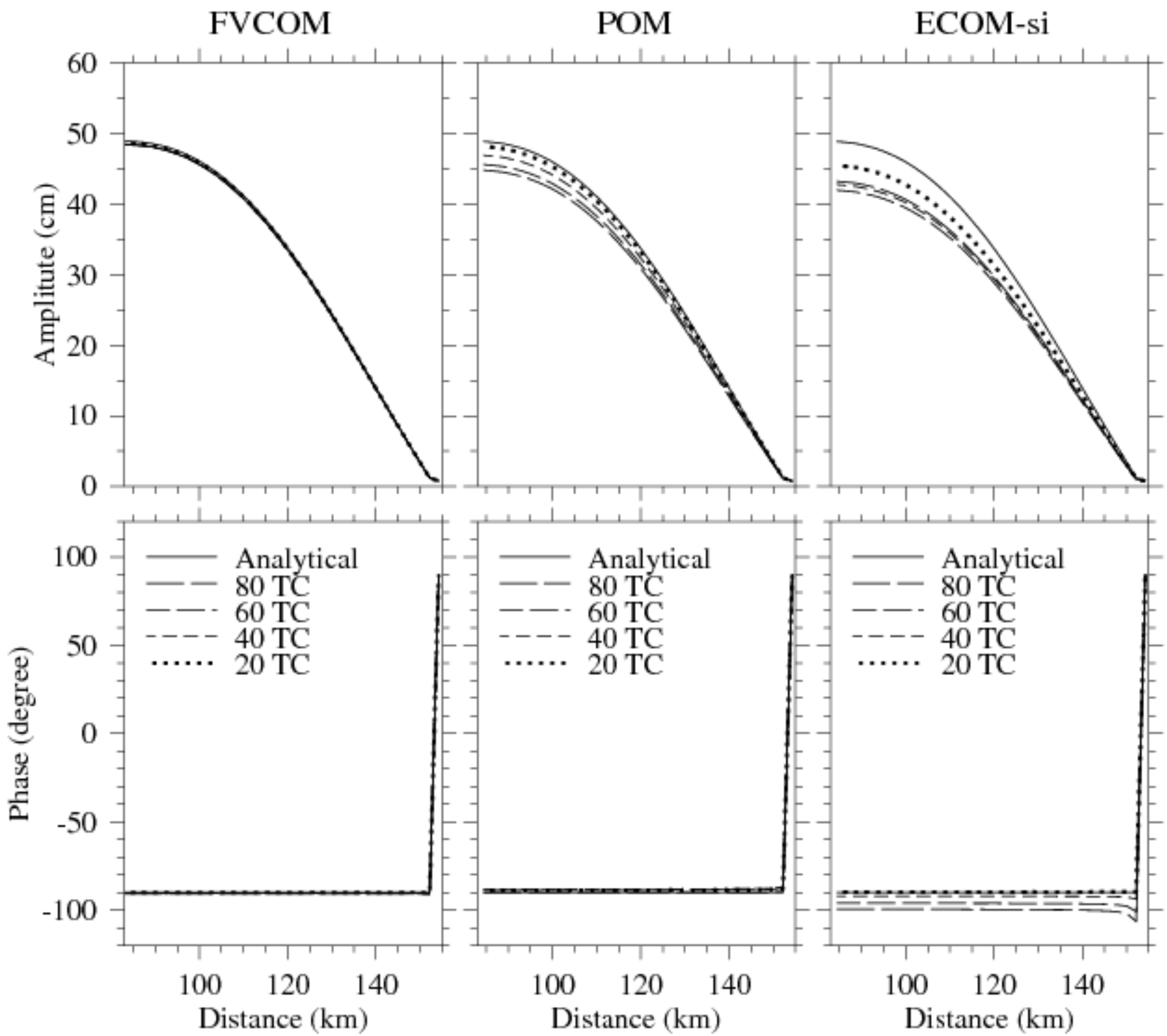
Near-resonance case

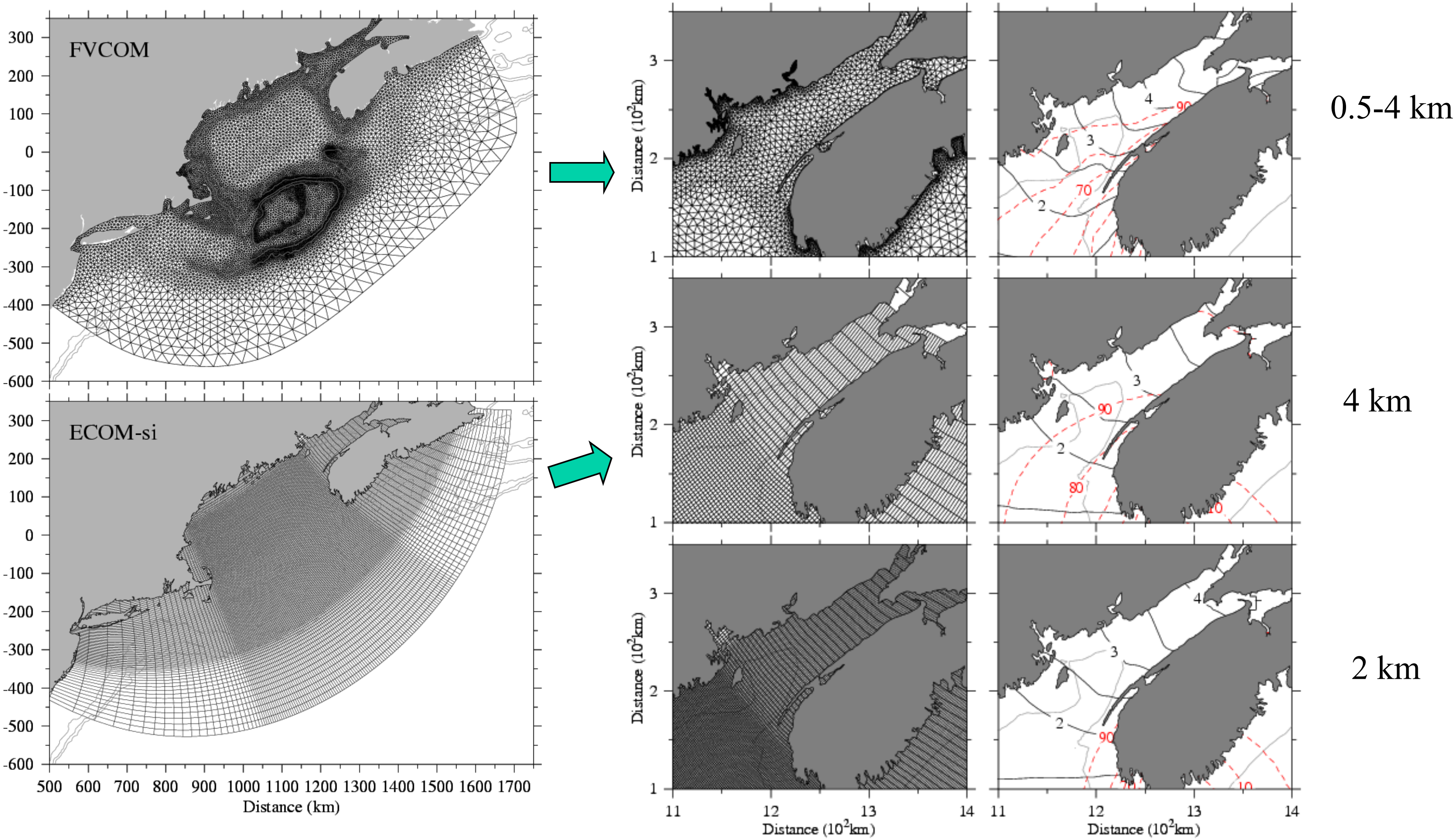


Near-resonance, 2 km (FVCOM), 1 km (POM&ECOM-si)

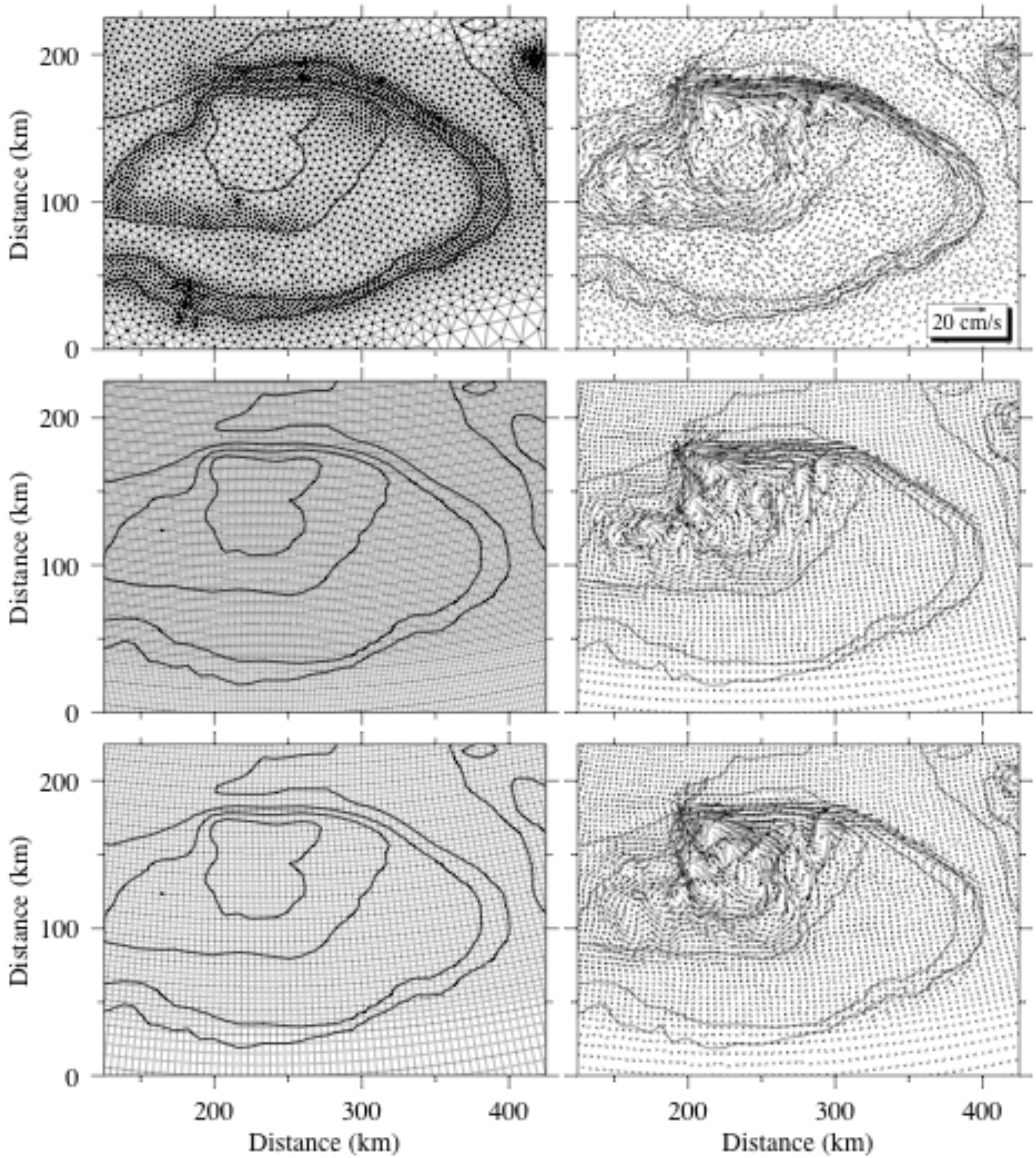


Near-resonance, 2km, Curvilinear

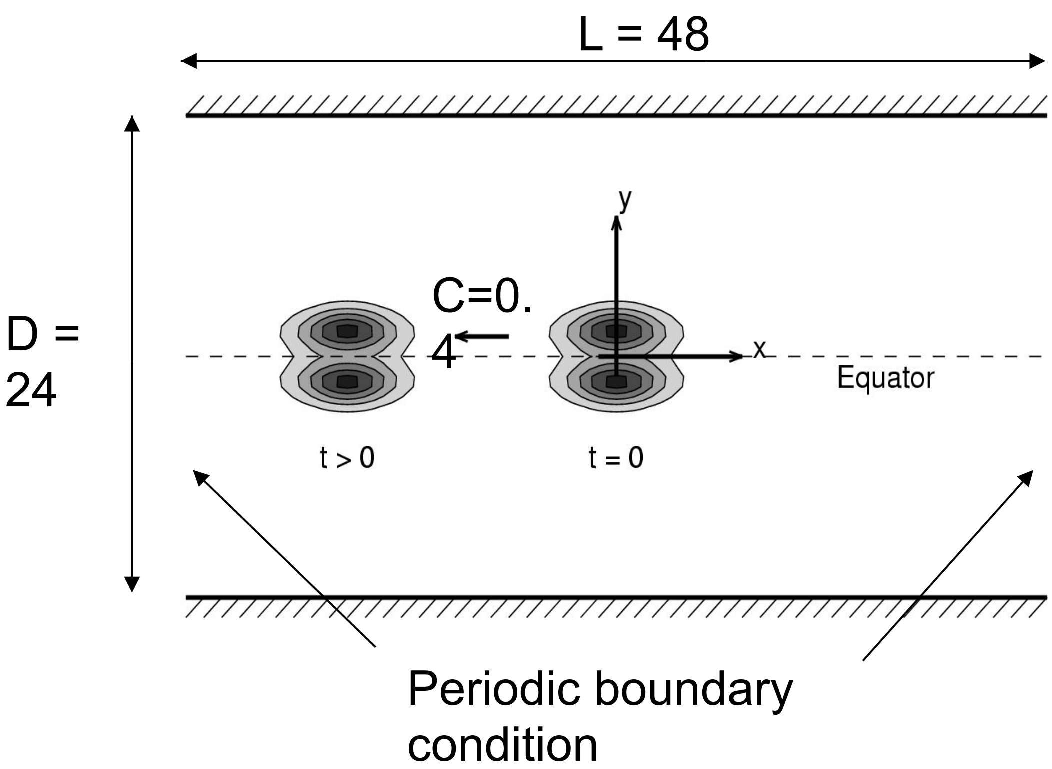



FVCOM
ECOM-si
Distance (km)
Distance (10^2km)
0.5-4 km
4 km
2 km

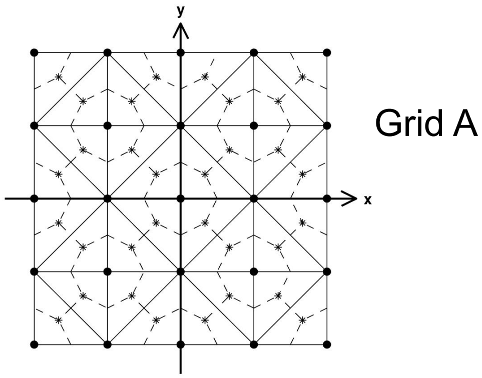
Slope topography fitting

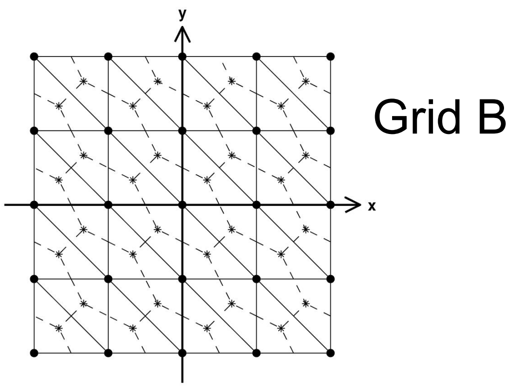


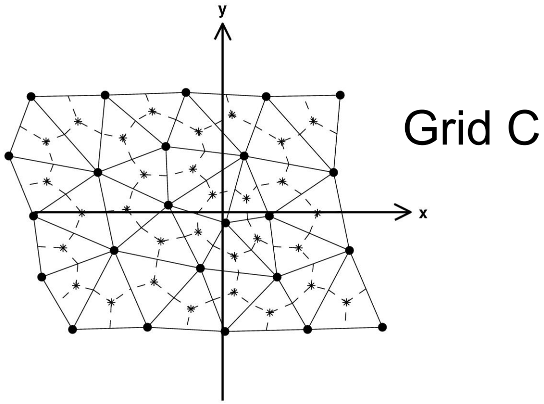
Equatorial Rossby Soliton

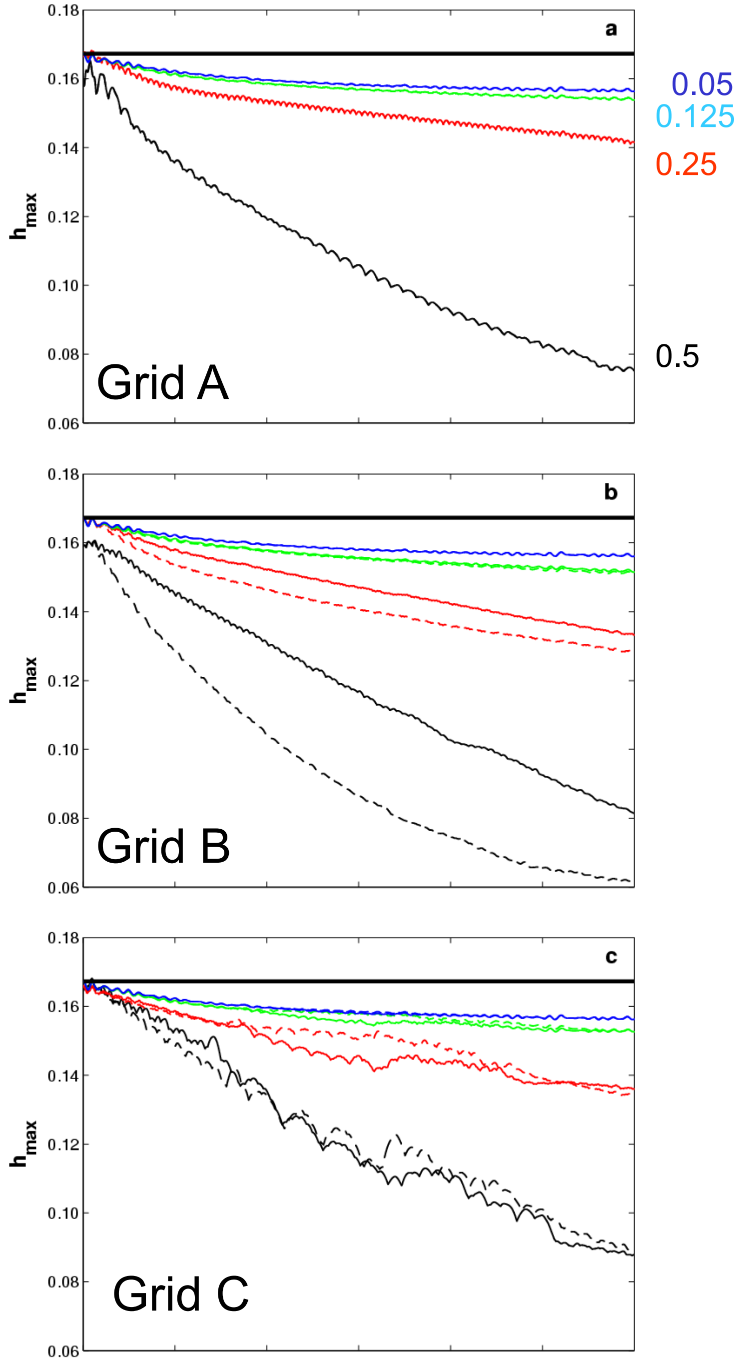


1. Nonlinear shallow water equation in equatorial β-plane
2. Inviscid flow
3. Asymptotic solutions available to zero and first orders (Boyd 1980,1985)









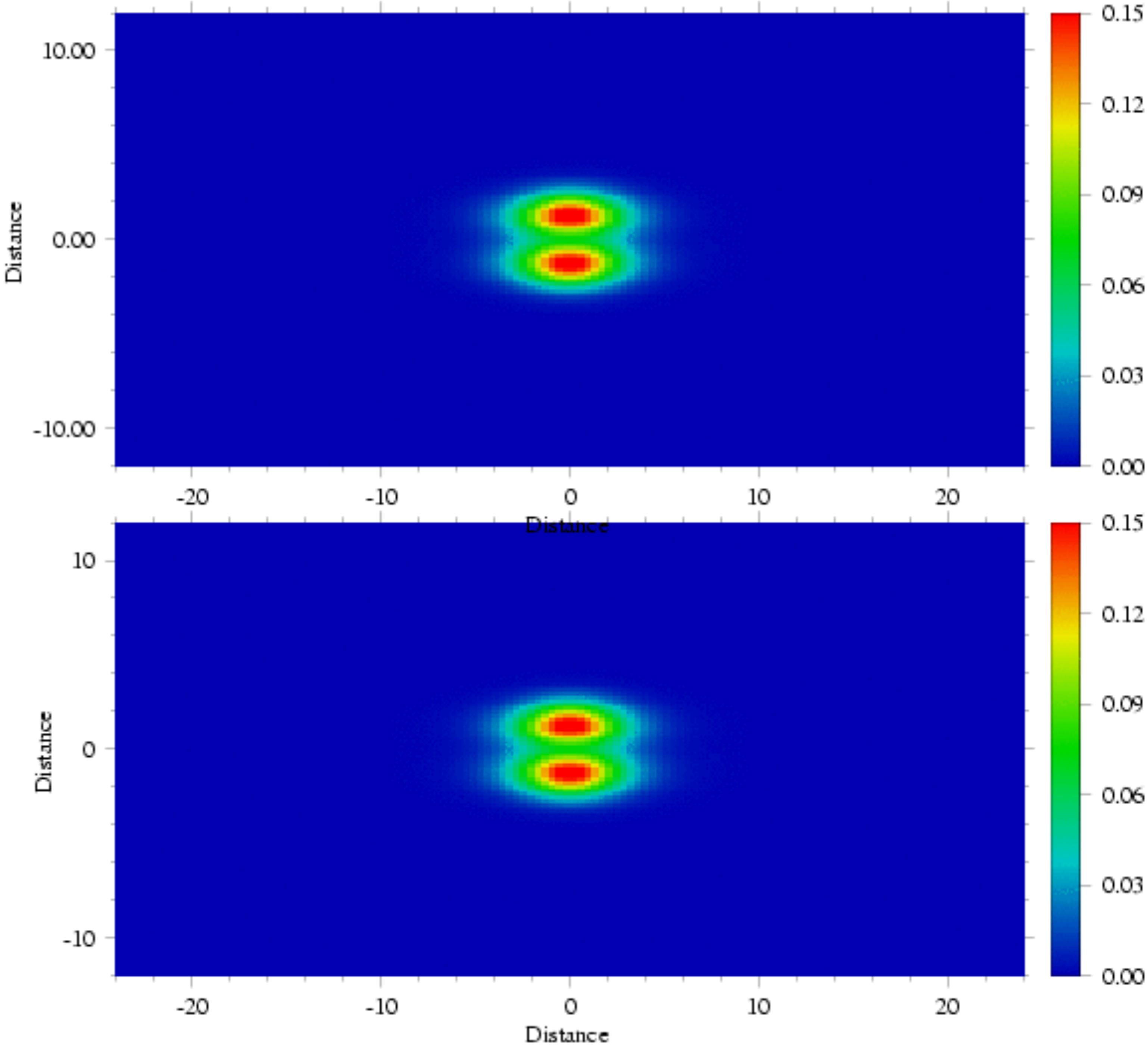
Δx (ND)	FVCOM (2nd)		ROMS (4th)		SEOM (7-9th)	
	h_n/h_t	C_n/C_t	h_n/h_t	C_n/C_t	h_n/h_t	C_n/C_t
0.5	0.472	0.917	0.884	1.088	0.923	0.98
0.25	0.846	0.984	0.926	0.993	0.929	0.99
0.125	0.92	0.984	0.923	0.986	0.937	0.989
0.05	0.935	0.983	0.936	0.983	0.915	0.98

h_n: Computed peak of the sea surface elevation at 120 units
h_t: Analytical peak of the sea surface elevation at 120 units
C_n: Computed average speed
C_t: Analytical averaged speed.

Comments:

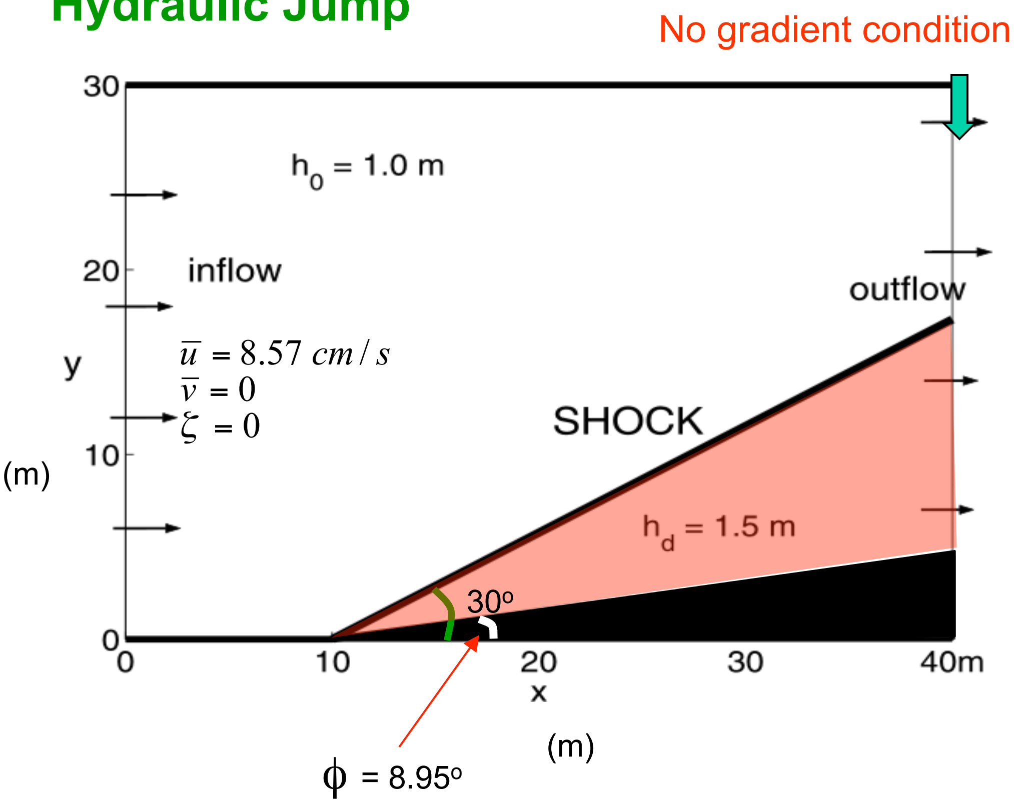
1. Analytical solution only represents the zero and 1st modes, while the numerical solution contains a complete set of higher order modes. This is not surprised to see numerical models can not exactly reach the analytical solutions.

2. FVCOM shows a fast convergence with increase of horizontal resolution.



10.00
0.00
-10.00
Distance
-20
-10
0
10
20
Distance
0.15
0.12
0.09
0.06
0.03
0.00
10
0
-10
Distance
-20
-10
0
10
20
Distance
0.15
0.12
0.09
0.06
0.03
0.00

Hydraulic Jump



Analytical solution:

Maximum sea level: $\zeta_{max} = 0.5$ m

Minimum sea level: $\zeta_{min} = 0$ m

Mean sea level: $\zeta_{mean} = 0.5$ m

Mean velocity: $\overline{u} = 7.956\ m/s$

Mean Froude #: $Fr = \dfrac{\overline{u}}{\sqrt{gD}} = 2.075$

Shock angle: $\alpha = 30^0$

Thickness: $\delta = 0$ m

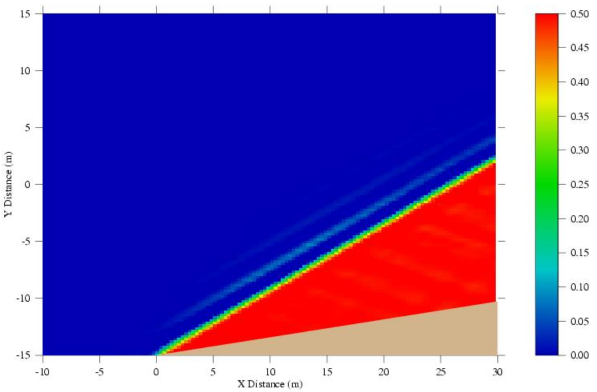
Mean deviation: $|dy| = 0$ m

Characteristics:

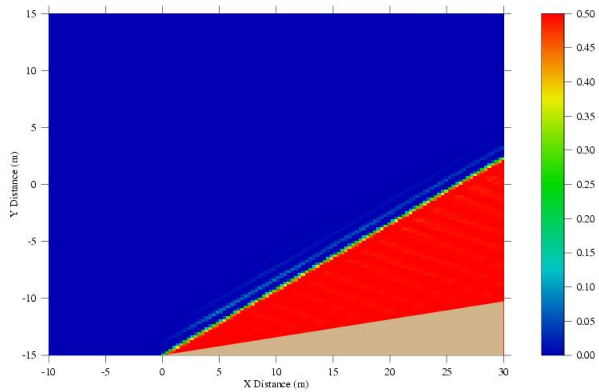
- Barotropic shallow water equations
- No rotation considered, i.e. f =0, β = 0
- Steady analytical solutions for u, ζ and the jump angle relative to the x axis.

The case with no horizontal diffusion: FVCOM quickly reaches steady status.

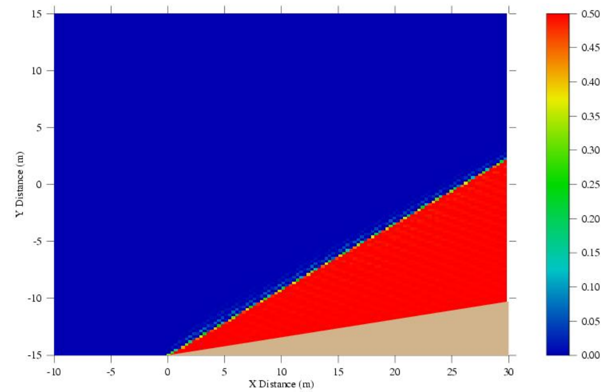
Δx = 0.5 m, Δt = 0.002



Δx = 0.25 m, Δt = 0.001

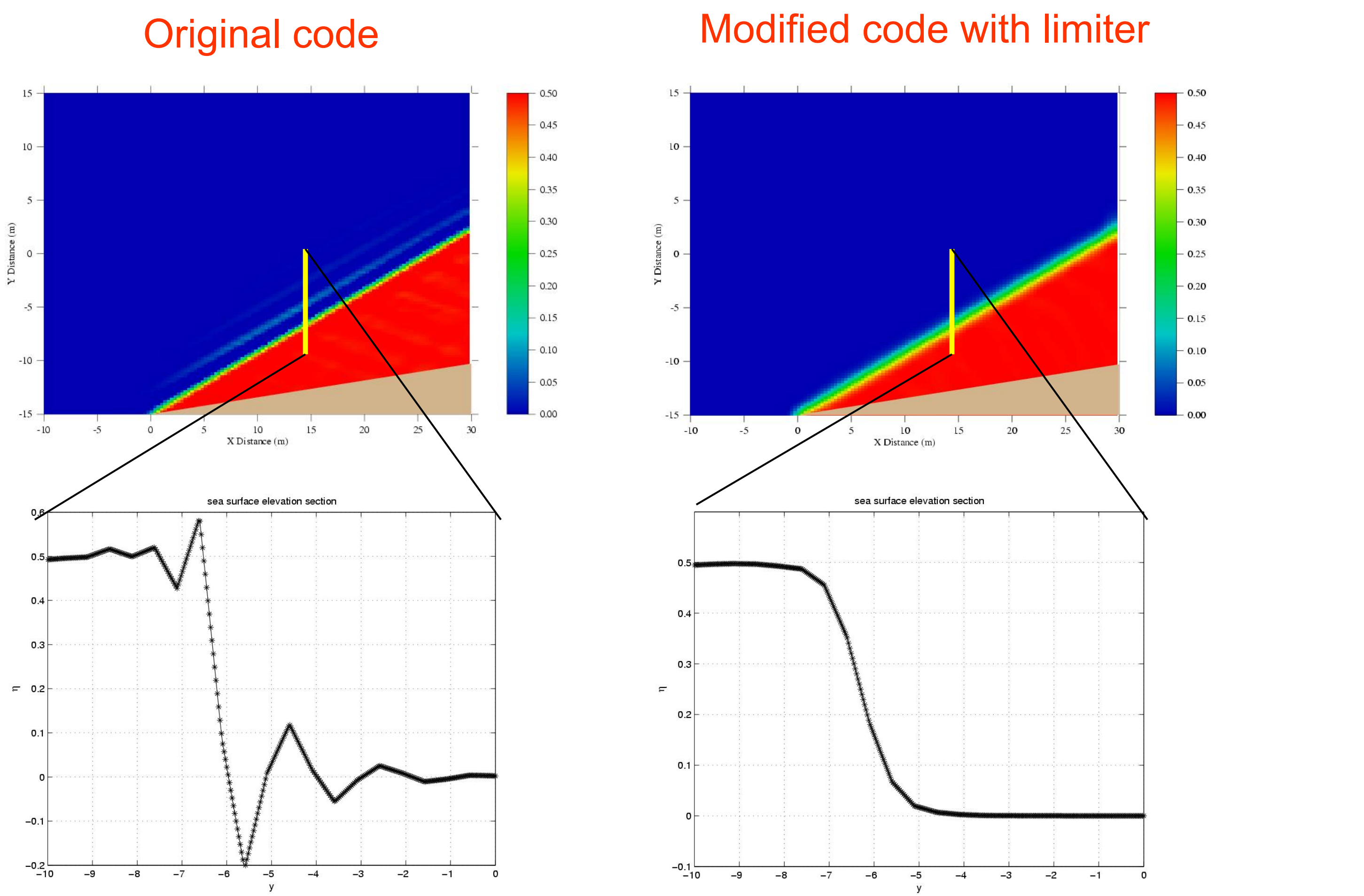


Δx = 0.125 m, Δt = 0.0005



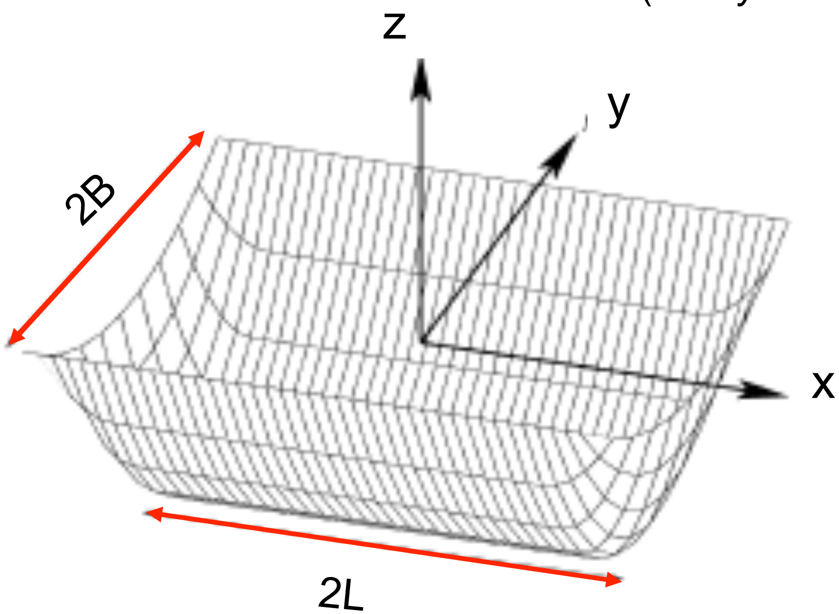
Model	grids	Δt	ζ_{max}	ζ_{min}	ζ_{mean}	$\bar{u}$	F_r	α	δ	$\|dy\|$
True			0.5	0	0.5	7.956	2.075	30	0	0
FVCOM	80 X 60	0.002	0.688	-0.269	0.5	7.949	2.072	29.952	0.111	0.305
	160 X 120	0.001	0.697	-0.268	0.499	7.951	2.073	30.030	0.063	0.151
	320 X240	0.0005	0.696	-0.272	0.5	7.951	2.073	30.029	0.037	0.076
ROMs	Reach an oscillatory solution without horizontal diffusion.									

Over shocking can be reduced by introducing a slope limiter method (Hubbard, *J. Comput. Phys.*, 1999).



3-Dimensional Wind-Driven Flow in an Elongated, Rotating Basin

Winant (*J. Phys. Oceano.* 2004)



Length: 2*L*; width: 2*B*, and bathymetry:

$$h = h_0\{0.08 + 0.92 * [X(x/L)(1-(y/B)^2)]\}$$

where *X(x)* is a function in the form of

$$X(x) = \begin{cases} 1, & |x| < 1-\Delta x \\ 1-[\dfrac{|x|-1+\Delta x}{\Delta x}]^2, & |x| \geq 1-\Delta x \end{cases}$$

Δx is a constant specified as 0.3% of the total length of the basin.

$$-L \leq x \leq L, -B \leq y \leq B$$

Governing equations:

$$\begin{cases} \nabla \cdot \vec{v} + w_z = 0 \\ f\vec{k} \times \vec{v} = -g\nabla\eta + K_m \dfrac{\partial^2 \vec{v}}{\partial z^2} \end{cases}$$

B.Cs:

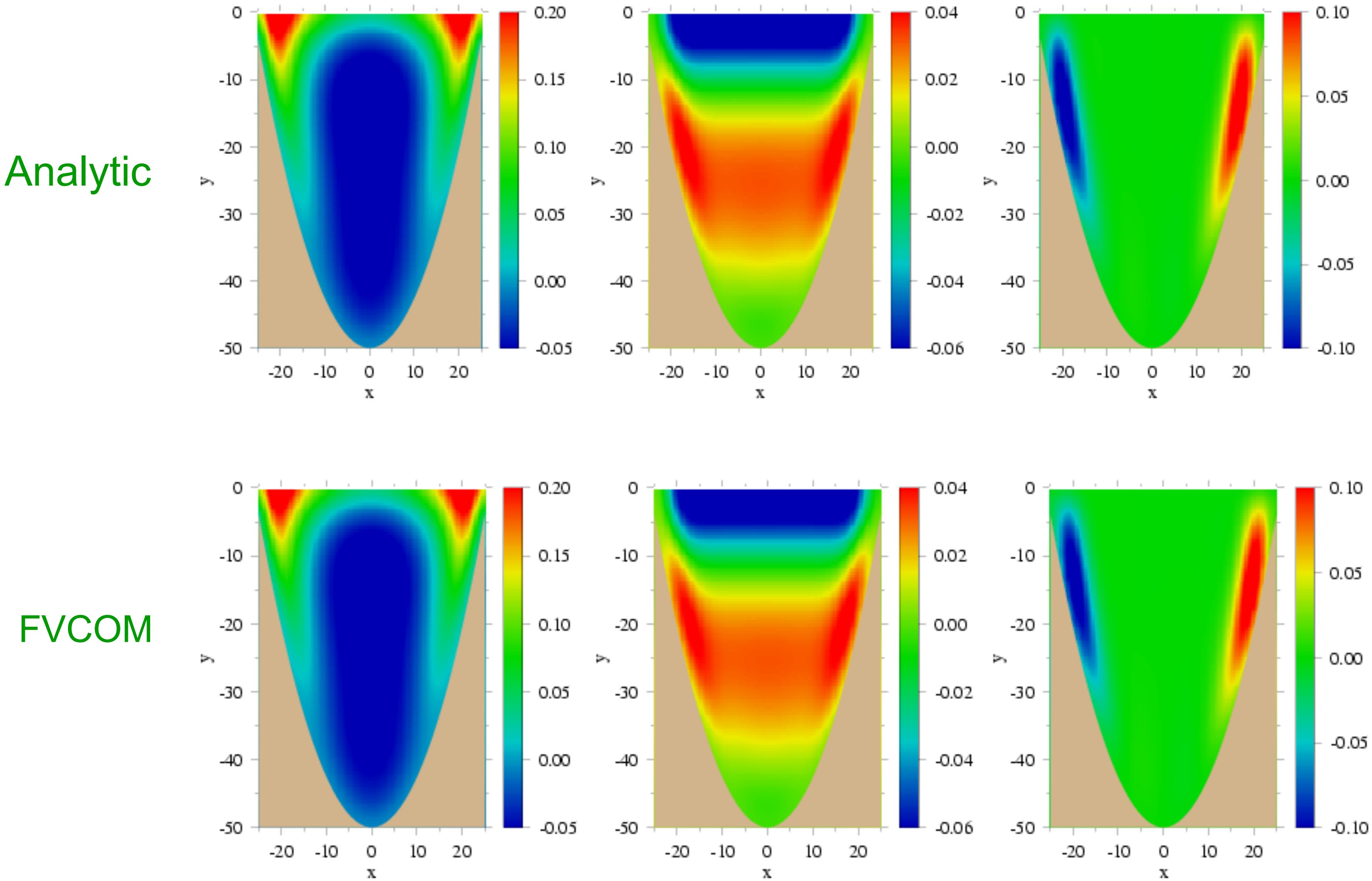
$$\begin{cases} \vec{v}_z = \dfrac{\vec{\tau}_s}{\rho K_m} \quad w = 0 \quad at\ z = 0 \\ \vec{v} = 0 \quad w = 0 \quad at\ z = -h \end{cases}$$

Steady status analytical solution for this linear equation system is given as:

$$u + iv = \frac{\sinh[\alpha(z+h)]}{\alpha\cosh(\alpha h)} - \frac{\eta_x + i\eta_y}{\alpha^2}[1 - \frac{\cosh(\alpha z)}{\cosh(\alpha h)}]$$

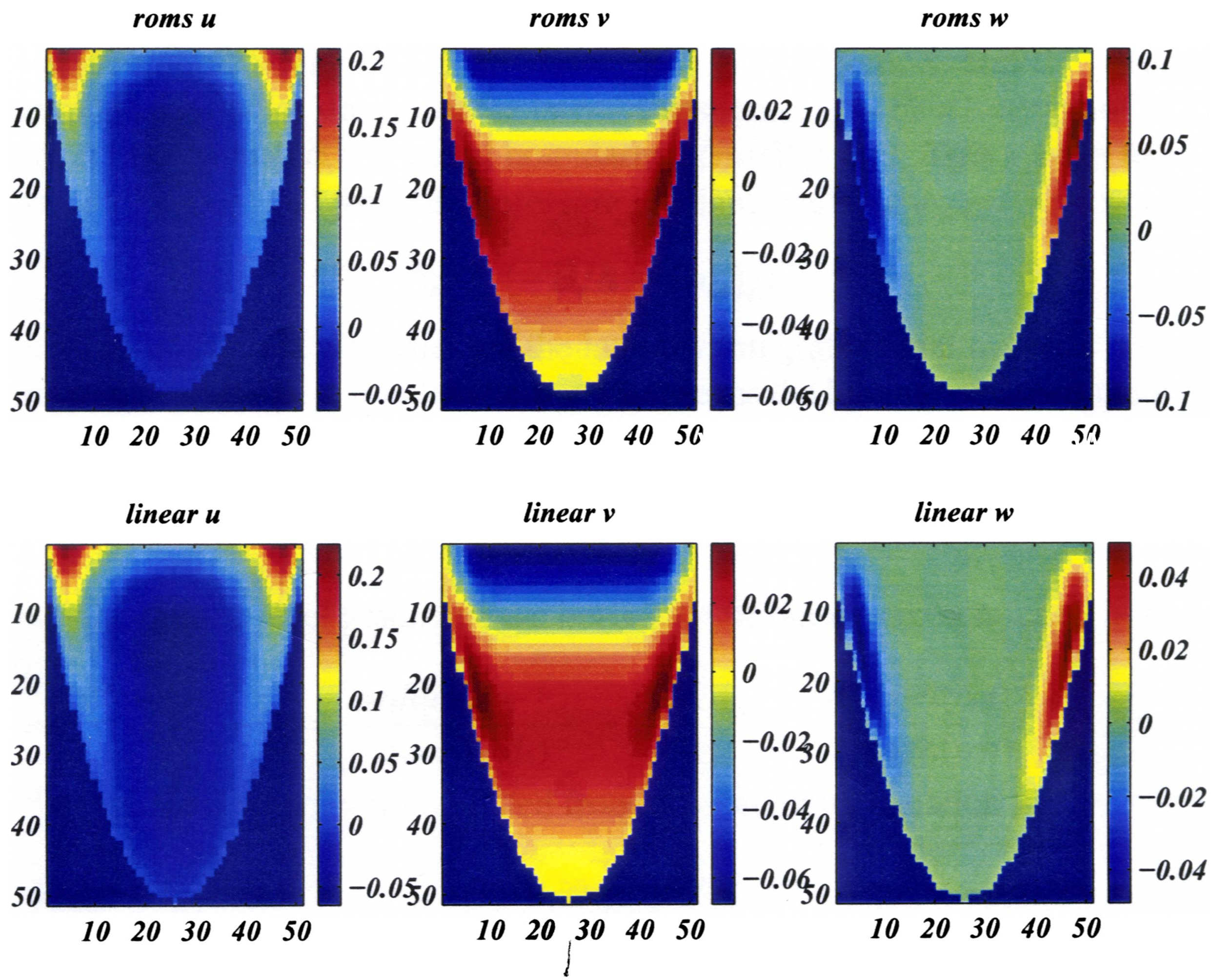
$$w = -\int v_y dz$$

where $\alpha^2 = 2i\delta^{-2}$ and $\delta = (2E)^{1/2}$ (*E*: Ekman number).


Analytic
FVCOM
y
x
0
-10
-20
-30
-40
-50
-20
-10
0
10
20
0.20
0.15
0.10
0.05
0.00
-0.05
0.04
0.02
0.00
-0.02
-0.04
-0.06
0.10
0.05
0.00
-0.05
-0.10

ROMs

Analytic



Be aware that ROMs underestimates *u* and overestimates *w* (color bar scales are different for analytical and ROMs' solutions). This figure is scanned from Winant's working note.